Mathematical Induction

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Abstract

First, we note that a proof by induction can point out its key step ; its initial case helps us find a starting point in a complicated environment. Second, a proof using the method of mathematical induction makes it easy to focus on the heart of the matter; in comparison, the proof without using mathematcal induction looks somewhat messy. Third, integration by reduction formula may help eliminate obstacles and automatically reduce the general case to the simplest one. Fourth, we discuss embedded levels of mathematical induction. Fifth, we discuss the art of using mathematical induction. Using mathematical induction is like managing a train: to save cost, we make the train run fewer times a day and increase the carload; to make the train run faster, we decrease the carload; the same idea of balancing carload applies to the effective use of mathematical induction. Sixth, we discuss summations and limitations of mathematical induction.

Keywords. Mathematical induction, embedded levels, how to effectively use mathematical induction, Peano's existence theorem, cardinal number, lattice, the Ascoli lemma, Plana's summation formula

1 A proof by induction can point out its key step

Example 1.1. (Gegenbauer's generalization of Poisson's integral)

The proof of Watson [7, p.50, (3)] given in Watson [7, p.50, l.11–l.18] requires the prerequisite given in Watson–Whittaker [6, §15.8]. It is difficult to locate its key step among the complicated formulas in Watson–Whittaker [6, §15.8]. In contrast, the following proof by induction highlights its key step: the recurrence formula $(n+1)C_{n+1}^{\nu}(t) = 2(\nu+n)tC_n^{\nu} - (1-t^2)\frac{dC_n^{\nu}(t)}{dt}$.

Proof. For $n = 0$, Watson [7, p.50, (3)] reduces to Watson [7, p.48, (4)].
$\frac{(-i)^{n+1}\Gamma(2\nu)(n+1)!(\frac{z}{2})^{\nu}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2\nu+n+1)} \int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}} C_{n+1}^{\nu}(t)dt$
$= \frac{(-i)^{n+1}\Gamma(2\nu)n!(\frac{z}{2})^{\nu}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2\nu+n)} \int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}} tC_n^{\nu}(t)dt$
$- \frac{(-i)^{n+1}\Gamma(2\nu)n!(\frac{z}{2})^{\nu}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(2\nu+n+1)} \int_{-1}^{1} e^{izt}(1-t^2)^{(\nu+1)-\frac{1}{2}} \frac{dC_{n+1}^{\nu}(t)}{dt}dt$ [Watson [7, p.50, l.−9]]
$= -J'_{\nu+n}(z) + \frac{(\nu+n)J_{\nu+n}}{z}$ [Watson–Whittaker [6, p.330, l.3, the first equation]; induction hypothesis]
$= J_{\nu+n+1}$ [Watson [7, p.45, (1) & (2)]]. □

Remark. It is troublesome to find a starting point or to go through details if we try to prove a complicated theorem from scratch. Especialy, it is difficult for a reader who follows this kind of proof to recognize its essence A good beginning is half the battle. The initial step of mathematical induction often provides such an appropriate starting point.

2 A proof by induction makes it easy to focus on the heart of the matter

Example 2.1. (Plana's summation formula)

Both Watson–Whittaker [6, p.145, l.−8–l.−3] and Guo–Wang [2, p.118, l.2–p.119, l.13] prove Plana's summation formula. The former proof looks somewhat messy. In contrast, the latter proof makes it easy to focus on the heart of the matter because it uses the method of mathematical induction.

3 Choosing the method of mathematical induction may help eliminate obstales and automatically reduce the case to the simplest one

A reduction formula establishes a recurrence relation which can reduce the form of an integral to a simper one. A recurrence relation is equivalent to an inductive step. If in a proof we choose to make a big jump without using reduction formula, we may encounter difficulties. This is because we may fail to predict the situation and control the key if we jump too far.

Example 3.1. (Stokes' formula)
$-\int_0^{\pi/2}\cos^{2n}\theta\ln\sin\theta d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\{\frac{\pi}{2}\ln 2+\frac{\pi}{4}(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n})\}$ [Watson [7, p.70, l.8]].

Complicated proof without using reduction formula.
Let $u=\ln\sin\theta$ and $dv=\cos^{2n}\theta d\theta$. Then
$v=\int\cos^{2n}\theta d\theta=\frac{\cos^{2n-1}\theta\sin\theta}{2n}+\frac{2n-1}{2n}(\frac{\cos^{2n-3}\theta\sin\theta}{2n-2}+\frac{2n-3}{2n-2}(\cdots+\frac{3}{4}(\int\cos^2\theta d\theta)\cdots))$
$=\frac{\cos^{2n-1}\theta\sin\theta}{2n}+\frac{2n-1}{2n}(\frac{\cos^{2n-3}\theta\sin\theta}{2n-2}+\frac{2n-3}{2n-2}(\cdots+\frac{3}{4}(\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2})\cdots))$.
$\int vdu=\int(\frac{\cos^{2n}\theta}{2n}+\frac{2n-1}{2n}(\frac{\cos^{2n-2}\theta}{2n-2}+\frac{2n-3}{2n-2}(\cdots+\frac{3}{4}(\frac{\theta\cot\theta}{2}+\frac{\cos^2\theta}{2})d\theta\cdots)))$.
$\frac{1}{2n}\int_0^{\pi/2}\cos^{2n}\theta d\theta=\frac{1}{2n}\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\frac{\pi}{2}=\frac{\pi}{4}\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\frac{1}{n}$.
$\frac{2n-1}{2n}\frac{1}{2n-2}\int_0^{\pi/2}\cos^{2n-2}\theta d\theta=\frac{2n-1}{2n}\frac{1}{2n-2}\frac{1\cdot3\cdots(2n-3)}{2\cdot4\cdots(2n-2)}\frac{\pi}{2}=\frac{\pi}{4}\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\frac{1}{n-1}$.
$\vdots$
$\frac{2n-1}{2n}\frac{2n-3}{2n-2}\cdots\frac{1}{2}\int_0^{\pi/2}\cos^2\theta d\theta=\frac{2n-1}{2n}\frac{2n-3}{2n-2}\cdots\frac{1}{2}\frac{\pi}{4}=\frac{\pi}{4}\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\frac{1}{1}$.
$\int_0^{\pi/2}\theta\cot\theta d\theta=\theta\ln\sin\theta|_0^{\pi/2}-\int_0^{\pi/2}\ln\sin\theta d\theta=\frac{\pi}{2}\ln 2$.
$\int_0^{\pi/2}vdu=\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)}\{\frac{\pi}{4}(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{1})+\frac{\pi}{2}\ln 2\}$
$=\frac{(2n)!}{2^{2n}(n!)^2}\{\frac{\pi}{4}(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n})+\frac{\pi}{2}\ln 2\}$. □

Simple proof using reduction formula.
Let $u=\cos^{2n-1}\theta\ln\sin\theta$ and $du=\cos\theta d\theta$. Then
$v=\sin\theta$ and $du=\cos^{2n-1}\theta\frac{\cos\theta}{\sin\theta}-(2n-1)\cos^{2n-2}\theta\sin\theta\ln\sin\theta$.
$\int vdu=\int(\cos^{2n}\theta-(2n-1)\cos^{2n-2}\theta\ln\sin\theta+(2n-1)\cos^{2n}\theta\ln\sin\theta)d\theta$.
$\int udv=uv-\int vdu$.

$\int_0^{\pi/2}\cos^{2n}\theta\ln\sin\theta d\theta = \frac{2n-1}{2n}\int_0^{\pi/2}\cos^{2n-2}\theta\ln\sin\theta d\theta - \frac{1}{2n}\int_0^{\pi/2}\cos^{2n}\theta d\theta.$
$-\int_0^{\pi/2}\cos^{2n}\theta\ln\sin\theta d\theta = \frac{2n-1}{2n}(-\int_0^{\pi/2}\cos^{2n-2}\theta\ln\sin\theta d\theta) + \frac{1}{2n}\int_0^{\pi/2}\cos^{2n}\theta d\theta.$
Thus, we can continue to reach $-\int_0^{\pi/2}\ln\sin\theta d\theta$ directly. □

Remark. In the first proof, we have to consider both $\int\cos^{2n}\theta d\theta$ and $\int_0^{\pi/2}\cos^{2n}\theta d\theta$. In contrast, in the second proof we need only consider $\int_0^{\pi/2}\cos^{2n}\theta d\theta$. The red parts in the first proof are difficult parts which disappear in the second proof. Note that the big jump in the first proof applies to $\int\cos^{2n}\theta d\theta$ alone so that it loses the connection with $\ln\sin\theta$.

4 Counting embedded levels of mathematical induction

If there is no other induction embedded in the inductive step of a mathematical induction, then the step is called a 1-level inductive step and the mathematical induction is called a 1-level induction. Assume that an $(n-1)$-level inductive step and an $(n-1)$-level induction are defined. If the main inductive step of a mathematical induction is embedded with an $(n-1)$-level induction, then the main step is called an n-level inductive step and the original induction is called an n-level induction.

Example 4.1. (The worst case for nonuniqueness)
Hartman [3, pp.18–23, §5] constructs an example using 4-level induction.
4-level induction: infinite induction on n [Hartman [3, p.18, l.17]]
3-level induction: infinite induction on k [Hartman [3, p.18, l.16]]
2-leve inductionl: infinite induction on i [Hartman [3, p.18, (5.2)]] for Case $n=0$; infinite induction on j [Hartman [3, p.20, l.11]] for Case $n+1$, where $n\geq 0$
1-level induction: finite (m) induction on i [Hartman [3, p.20, (5.15)]] for Case $n+1$, where $n\geq 0$

Remark 1. The equality given in Hartman [3, p.21, (5.21)] should have been replaced with
$v_i(t) = u_i(t)\sin^2 2^n\pi(t-c) + u_{i+1}\cos^2 2^n\pi(t-c).$

Remark 2. The equality given in Hartman [3, p.21, (5.24)] should have been replaced with
$v_i'(t) = u_i'(t)\sin^2 2^n\pi(t-c) + u_{i+1}'\cos^2 2^n\pi(t-c) + 2^n\pi(u_i - u_{i+1})\sin^2 2^{n+1}\pi(t-c).$

Remark 3. The two equalities given in Hartman [3, p.21, (5.25)] should have been replaced with
$|v_i'| \leq M_n + 2^n\pi\frac{d_n}{m}, |v_i''| \leq M_n + (2^{n+1}+2^{2n+1}\pi)\pi\frac{d_n}{m}.$

Remark 4. The two equalities given in Hartman [3, p.21, l.12–l.13] should have been replaced with
$v_i - u_i = (u_{i+1}-u_i)\cos^2 2^n\pi(t-c),$
$v_i - u_{i+1} = (u_i - u_{i+1})\sin^2 2^n\pi(t-c).$

Remark 5. The second inequality given in Hartman [3, p.21, (5.26)] should have been replaced with
$|v_i - u_h| \leq (1+2^n\pi)\frac{d_n}{m}.$

Remark 6. The inequality given in Hartman [3, p.21, (5.27)] should have been replaced with
$(2^{n+1}+2^{2n+1}\pi)\pi\frac{d_n}{m} < \frac{\varepsilon_{n+1}}{3}.$

Example 4.2. (Lattices)
One may define the n-dimensional lattice $\{(k_1,\cdots,k_n)|k_i\in\mathbb{N}, 1\leq i\leq n\}$ using an n-level induction and

define the $\aleph_0$-dimensional lattice $\{(k_1, \cdots, k_i, \cdots) | k_i \in \mathbb{N}, i \in \mathbb{N}\}$ using an $\aleph_0$-level induction.
Remark. The use of an $\aleph_0$-level induction is the most sophisticated and cumbersome construction by mathematical induction.

5 How to effectively use mathematical induction

A wise use of mathematical induction may help us avoid complications, make proofs more constructive, effective, and meaningful. Only simple tasks and a minimum amount of work should be performed within an inductive step. The rule of thumb is that in a proof we should take everything unnecessary out of an inductive step if possible. In other words, mathematical induction should be applied to only the inevitable part of a proof.

In computer programs mathematical induction involves recursion. In order to save the cost of memory and operations, we should perform only simple tasks and a minimum amount of work within an inductive step.

Example 5.1. (Peano's existence theorem)
Let us compare the proof of Coddington–Levinson [1, p.6, Theorem 1.2] with that of Hartman [3, p.10, Theorem 2.1]. First, the former provides an error estimate [Coddington–Levinson [1, p.3, theorem 1.1]], while the latter fails to do. Second, Coddington–Levinson [1, p.4, Fig. 1] uses finite induction to construct φ, while Hartman [3, p.10, (2.1)] uses finite induction on n to expand the domain $[t_0, t_0 + n\varepsilon]$ of y_ε. The inductive step in the former construction involves drawing a line and finding the intersection with a vertical line. They are simple tasks and can be done in a finite steps. In contrast, the inductive step in the latter construction involves integration which is an infinite process. Third, based on the definition of limit, we must use induction on n when using $\lim_{y\to\infty} y_n$. All the natural numbers n except a finite number of them have to be taken into account. Let us consider the main inductive steps involved in the above two proofs. In order to derive Coddington–Levinson [1, p.7, (1.7)] from Coddington–Levinson [1, p.6, (1.6)], we only need to note that $|\Delta_n(t)| \leq \varepsilon_n$. In contrast, deriving Hartman [3, p.9, (1.5)] from Hartman [3, p.10, (2.1)] requires more operations:
$|f(t, y_{\varepsilon(n)}(t-\varepsilon(n))) - f(t, y(t))|$
$\leq K|y_{\varepsilon(n)}(t-\varepsilon(n)) - y(t)|$
$\leq K[|y_{\varepsilon(n)}(t-\varepsilon(n)) - y_{\varepsilon(n)}(t)| + |y_{\varepsilon(n)}(t) - y(t)|]$
$\leq K[M\varepsilon(n) + |y_{\varepsilon(n)}(t) - y(t)|]$.
Fourth, both proofs use the Ascoli lemma [Coddington–Levinson [1, p.6, -12]; Hartman [3, p.11, l.2]]. In view of the above considerations, Levinson's proof avoids complications and focuses on the essence of the theorem.

Remark.

Table 5.1: Train vs. Mathematical induction

Train	Mathematical induction
To save cost, let a train run once instead of twice. Increase carloads.	To avoid unnecessary abuse, we should not use mathematical induction twice if the proof can be done by using it once. Let each inductive step do more work.
In order to make the train run fast, load a minimum amount of cargo on each car.	For a computer, in order to increase its efficiency and save its cost of memory and operations, we should perform a minimum amount of work within an inductive step.
We should adjust the amount of cargo on each car so that it meets the practical requirement of reducing the number of trips and increasing its speed.	We should adjust the amount of work in each inductive step so that it meets the practical requirement of reducing the number of times of using mathematical induction and increasing computer efficiency.

6 Miscellaneous notes about mathematical induction

(1). (Summation of finite terms) Mathematical induction can be used to describe the set of natural numbers. Finite induction can be used to describe a finite number of natural numbers. The cardinal number of index set of a summation can be finite. A summation should not just list a few first terms; it should indicate the last term. Notations should not be awkward; they should be able to reveal simple relations and should be easy to handle. Compare the equalities given in Hobson [4, p.107, (7); p.108, (10)] with those given in Watson [7, p.33, l.17–l.18].

(2). (Limitations of mathematical induction) In essence, the method of mathematical induction is not creative. It is only good for organizing a proof. The proof given in Watson [7, p.33, l.12–p.34, l.-10] provides and proves the expansion given in Watson [7, p.34, (1)]. In order to avoid the elaborate analysis in the above proof, Lommel proves the same formula using mathematical induction [Watson [7, p.34, l.-6–p.35, l.8]]. Note that the latter proof alone cannot provide the form of expansion.
Remark. Mistakes in a textbook can determine a reader's academic level: inability to discover them, ability to discover them, or ability to correct them. The argument given in Watson [7, p.35, l.6-l.7] should have been corrected as follows:

Correction. By Watson [7, p.16, (4)], $|J_n(z)| \leq \frac{|\frac{z}{2}|^n}{n!}\exp(\frac{|z|^2}{4})$.
$\sum_{n=0}^{\infty}\frac{(m+n)!}{n!}|J_{m+2n}(z)| + \sum_{n=0}^{\infty}\frac{(m+n)!}{n!}|J_{m+2n+2}(z)|$
$\leq |\frac{z}{2}|^m \exp(\frac{|z|^2}{4})\sum_{n=0}^{\infty}\frac{(|\frac{z}{2}|^2)^n}{n!} + |\frac{z}{2}|^{m+2}\exp(\frac{|z|^2}{4})\sum_{n=0}^{\infty}\frac{(|\frac{z}{2}|^2)^n}{n!}$. □

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