Mathematical Problems that Physicists Fail to Solve

Li-Chung Wang

December 24, 2021

Abstract

There exist many mathematical statements in physics that physicists constantly use but fail to prove. Now I try to discuss these problems in the following sections:
§1. The relationship between the number of constraints and degrees of freedom

Keywords.

1 The relationship between the number of holonomic constraints and degrees of freedom

A system of N particles free from constraints, has $3N$ independent coordinates or degrees of freedom. If there exist holonomic constraints, expressed in k equations in the form (1.37), then we may use these equations to eliminate k of the $3N$ coordinates, and we are left with $3N-k$ independent coordinates, and the system is said to have $3N-k$ degrees of freedom [Goldstein [2, p.13, l.-18–l.-14]]. Suppose$k=1$ and the constraint has the form $f(x_1,x_2,\cdots,x_{n-1},x_n)=0$. Let $x_n'=f(x_1,x_2,\cdots,x_{n-1},x_n) \quad (*)$. Then the coordinates $(x_1,x_2,\cdots,x_{n-1},x_n')=(x_1,x_2,\cdots,x_{n-1},0)$ has $n-1$ degrees of freedom. We may solve $(*)$ to obtain $x_n=g(x_1,x_2,\cdots,x_{n-1},x_n')$.

proof of the statement in red. It suffices to prove the case $k=1$.
I. Case I: the constraint is linear. We may assume the constraint is homogeneous by a coordinate change. The proof for this case is given in Jacobson [3, vol.II, p.45, Theorem 4]. For example, we may use this theorem of Jacobson's to prove $\dim M^R \cap M^S = n^2-1$ [Chevally [1, p.8, l.-9–l.-8]].
II. A special case (the case of exponential function $\exp:\mathfrak{M}_n(\mathbb{C})\to GL(n,\mathbb{C})$)
Chevally [1, p.8, Proposition 5] uses the uniqueness of dimension of a topological manifold to prove
(a). $\dim M^S = \dim SL(n,\mathbb{C})$.

$\exp : M^S \to SL(n,\mathbb{C})$		
Elements in the domain & their corresponding images	$A = (a_{ij})$	$\exp A = (x_{ij})$
The constraints	$\mathrm{Sp}\, A = \sum_i a_{ii} = 0 \quad (1)$	$\det(x_{ij}) = \det(\exp A) = \exp(\mathrm{Sp}\, A)$ [Chevally [1, p.6, Corollary 1]]$= 1 \quad (1^*)$
Observations	The number of contraints are the same; the relation between (1) and (1^*) is explicit [Chevally [1, p.5, Proposition 2]]; if we replace a_{ij} in (1) with x_{ij} , then (1) becomes (1^*), and vice versa.	

(b). $\dim M^{sh} = \dim U(n)$.

$\exp : M^{sh} \to U(n)$		
Elements in the domain & their corresponding images	$A = (a_{ij})$	$\exp A = (x_{ij})$
The constraints	${}^t\alpha = -\bar{\alpha} \quad (2)$	$\exp({}^t\alpha) = \exp(-\bar{\alpha}) \Leftrightarrow {}^t(\exp\alpha) = (\overline{\exp\alpha})^{-1} \Leftrightarrow (\exp\alpha)^* = \overline{\exp\alpha} \quad (2^*)$
Observations	The number of contraints are the same; the relation between (1) and (1^*) is implicit, but Chevally [1, p.7, Proposition 4] allows us to predict the general case; if we replace a_{ij} in (2) with x_{ij} , then (2) becomes (2^*), and vice versa [Chevally [1, p.8, Lemma 1]].	

III. The general case (the local diffeomorphism case [Lee [4, p.79, Theorem 4.5]; a coordinate change refers to this case]).

Let M, N be n-dimensional smooth manifolds, F be a local diffeomorphism from M onto N, and $G(p) = 0$ be a smooth constraint in $p \in M$. Locally, we may treat M as T_pM, F as dF, and $G(p)$ as its linear approximation. Then the general case is reduced to Case I. □

References

[1] Chevally, C.: *Theory of Lie groups I*, Princeton: Princeton University Press, 1946.

[2] Goldstein, H., Poole, C., & Safko, J: *Classical Mechanics*, 3rd, ed.,New York: Addison-Wesley, 2001.

[3] Jacobson, N.: *Lectures in Abstract Algebra*, 3 vols., Princeton: Van Nostrand, vol. 1 (1951), vol. 2 (1953), vol.3 (1964).

[4] Lee, J. M.: *Introduction to smooth manifolds*, 2nd ed., New York: Springer , 2013.

Mr. Li-Chung Wang is the author of the following website about the philosophy of mechanics:
http://www.lcwangpress.com/physics/main.html.
Address: 7th Floor, #21 Lane 267, Xi-zhou Street, Chungli, Taiwan, ROC.
E-mail:lcwangpress@yahoo.com