Mathematical Methods

Li-Chung Wang

December 10, 2011

Abstract

Mathematical training is to teach students the methods of removing obstacles, especially the standard ones that will be used again and again. The more methods one has learned, the higher one’s skill and the more chances one may do some creative works in mathematics. In this paper, we try to classify all the mathematical methods. Mathematical methods are too many to count. They look too diversified to manage. If we attempt to classify them by luck, it would be difficult as though we were lost in a maze and tried to get out. However, if we carefully study a method’s origins, development and functions, we may find some clues about our task by following along their texture. Studying methods helps us analyze patterns. In order to make a method’s essence outstanding, the setting must be simple. Using a method is like producing a product: A method can be divided into three stages: input, process, and output. Suppose we compare Method A with Method B. If the input of Method B is more than or equal to that of Method A and if both the process and the output of Method B are better than those of Method A, then we say that Method B is more productive than Method A. Here are some examples. Cases when the input is increased: The method given in the general case can be applied to specific cases. However, specific cases contain more resources, there may be more effective methods available for specific cases. Our goal is to seek the most effective method in each case. Strategies for improving the output: When we formulate a theorem, the conclusion should be as strong as possible. When we seek a solution, the solution should be specific and precise. The execution of the solving plan should be finished perfectly and the solution should be expressed in closed form if possible. Strategies for improving the process: When we introduce a definition, the definition should be accessible in a finite number of steps; the target should be reached quickly, directly and effectively. When we present a proof or theory, we should avoid repetition, organize it in a consistent and systematic way; make it leaner and simpler. The qualities of process can be roughly divided into following categories: Accessibility, functions (reduction [reduction from a large number of cases to a smaller one], analogy, effectiveness, simplicity, directness, flexibility, fully utilizing resources, hitting multiple targets with one shot, avoiding contradictions, avoiding repetition and unnecessary complications, expanding the scope of application without loss of efficiency), structurization (insightfulness, essence, flowcharts or networks). Structurization may help us systematically operate and deeply understand the complicated cases. Like two neighboring colors in a rainbow, there is no clear dividing line between two qualities. The method of weakening a hypothesis is a method of reducing the input for a given output. The concept of almost everywhere can often ruin the delicacy of the method of weakening a hypothesis. For physical methods, we discuss physical interpretations and physical proofs. As we go to a more advanced level and broaden our consideration, new physical meanings of mathematical equations continue to develop and meanings of equations become richer and more delicate. Nonetheless the meanings in older theories are still well-preserved in a newer theory. Guided by a theorem’s physical meaning, one may develop a better strategy to prove it. The ideal physical proof is the one each of whose step has a pertinent physical interpretation. The development of physical methods shows the tendency toward such an ideal. For example, the mathematical formulation of the second law of thermodynamics leads to a criterion for integrability of Pfaffian forms. Sometimes, a mathematical proof requires advanced
and complicated knowledge and a long argument; we may easily get lost. In contrast, a physical proof may lay bare the key idea with one penetrating remark. A physical proof is usually more direct than a geometric proof. Physical and geometric proofs provide richer meanings, insights, and interesting stories than analytic proofs. A proper physics model can be a natural guide to the study of PDEs. Improvements of classical methods: a remedy rather than a thorough revamp is all we need; this introduction mode based on needs may make the key to improvement most outstanding. A hyperbolic paraboloid has center at \((\infty, \infty, \infty)\) because Cartesian coordinates lack ability to distinguish infinities of all directions; if we use spherical coordinates instead, there will be no common midpoint for the chords through the origin. If the set of centers is empty and we allow a point involving \(\infty\) to be its element due to a tool abuse, then all the theorems to which the false existence of elements leads will be meaningless. Only after a construction is tailored to our needs can it solve the problem effectively. An index set must be chosen properly: one more candidate would be too many and one less would be too few. The Dirac delta generalized function should not be treated as a function; we provide an easy way to bridge the gap between a function and a generalized function. A good theorem should provide complete information. The Ritz method is an effective tool for studying Sturm–Liouville Problems. In order to effectively solve a problem, we must quickly understand the circumstance with the minimum effort, and then directly attack the heart of the matter. Find extrema with subsidiary conditions. Compare physics proofs with mathematics proofs. Distribution theory is a new theory that we create to avoid the contradiction that the domain of a function contains a point whose function value cannot be defined. How do we deal with a problem that may easily cause us to commit errors?

When studying a generalized definition, we should understand its primitive version, its entire process of revolution, and the reason for the necessity of generalization; if we proceed directly toward the most general version in axiomatic approaches, its setting usually requires a more strange language and less familiar structures which may blur the essential idea, and the algorithm to check the definition usually becomes less effective; thus, an improper approach to generalized definitions may easily lead to an empty formality and make it difficult for us to see the advantages of generalized definitions over the primitive version; providing several non-trivial examples alone is not enough.

“Using formulas in a table without care may easily result in mistakes. One is under the impression that once the solution form is obtained, the actual solution is determined. This is not so. If the resulting function is multivalued and the formula fails to indicate which value to choose, then the formula would be useless. One should find a delicate method to determine the correct value. If one uses such a unfinished formula in a proof, then the proof would be incorrect. Such a mistake is often difficult to detect.”

“How do we detect errors in a textbook? When I find an error, the first response is usually to refuse to accept this fact and try to rationalize the opposite viewpoint. After all, there are many authors who have not found it incorrect after copying it. Nevertheless, I try to remember this odd experience so that I can easily find a reason when a problem occurs afterwards. However, this “rationalization” actually conceals a mistake. The reason why we fail to detect an error is that we have not gone far enough to forsee its consequences. Errors cannot withstand tests. Soon or later they will be detected. Even if an error may not be detected at the first checkpoint in application, it can hardly survive at the second one. If we consider an error true, then the world would fall into pieces as if Pandora’s box were opened. I became so frustrated that I had to choose the other option: the error is not necessarily true. Then I found a counterexample. I could omit some details and still make this paragraph logical, but this would destroy the evidence of true experience and eliminate the track of the natural thought for solving a problem.”

“Contour integrals for special functions: I. When we deal with a contour integral for a special function, all we have to do is to choose a point on the contour and assign a possible value to its argument. II. The only purpose of detailed discussion about branch points is to tell us that if we want to choose a point and its argument properly to facilitate calculations, we must consider branch points first.”

“Tying up loose ends”

“The finishing touch: Providing a solution to a problem alone is not enough; the author should tell the readers from where the solution comes. This way can bring the readers to an advantageous point for a bird’s-eye view of the circumstance.”

“Musket to kill a butterfly: Example [Differentiating under an integral]. The modern method attacks

2
directly toward the goal by using theorems flexibly. A complex measure need not distinguish a compact integral contour from a noncompact one. A single proof is good enough for dealing with both compact and noncompact cases. Furthermore, the proof is free from complex analysis except for using the definition of analytic functions. In contrast, the classical method must follow a formal, tedious, and inflexible procedure. In order to ensure the finiteness of a contour integral, the Borel measure must distinguish a compact integral contour from a noncompact one. In fact, in order to include the case of noncompact integral contour, the modification and supplement have to use almost all the theorems in complex analysis and, thus, lead to unnecessary complications.”

“Grasping the overall situation: Hypergeometric functions and confluent hypergeometric functions are closely related. We must build paths between the two topics as many as possible. When we discuss confluent hypergeometric functions, of course, we have to include their characteristic properties. Furthermore, for each property, we should find its corresponding property in hypergeometric functions, treat the latter as a motivation of the former and use the latter to prove the former. Just because of the complicated circumstance, we should give a rigorous proof rather than touch it lightly. Otherwise, the discussion is incomplete. Sneddon [91, p.32, l.1–l.18] sets a good example for discussing confluent hypergeometric functions.”

“Linear transformations of the hypergeometric function: the general equation of Fuchsian type having three regular singularities vs. the standard hypergeometric equation; eliminating repetitions in an overestimate vs. the counting based on the correspondence between solution pairs and regular singularities; Guo–Wang [46, p.141, (4) & (5)] can be derived from Guo–Wang [46, p.140, (2) & (3)] by inspection [Watson–Whittaker [108, p.207, (I) & (II)]]; calculations vs. inspection; Lebedev [65, §9.5] shows that Lebedev [65, p.249, (9.5.8) & (9.5.9); p.250, (9.5.10)] all follow from Lebedev [65, p.249, (9.5.7); p.247, (9.5.1) & (9.5.2)]; based on the list of linear transformations given in Lebedev [65, p.246, l.1–l.14], the discussion given in Lebedev [65, §9.5] is complete; the formula given in Watson–Whittaker [108, p.289, l.1–l.5] and the one given in Watson–Whittaker [108, p.291, l.1–l.5] are proved the hard way because they both use the contour integrals of Barnes’ type [p.286, l.1–l.7–p.287, l.3; p.289, l.1–l.18–l.17] and the residue theorem; in fact, we can still prove Lebedev [65, p.247, (9.5.1) & (9.5.2); p.248, (9.5.4); pp.249–250, (9.5.7)–(9.5.10)] without using any integral representation.”

“Methodical solutions: First, consider a differential equation of a special type. If its integral solution is based on guess, luck, and trial-and-error, we do not know from where the integrand comes, and the only way to justify the solution is by substitution, then this underdeveloped solution cannot be considered a methodical solution. Suppose the same equation also belongs to the wider class of equations of Laplacian type. In contrast, its integral solution can be built by a systematic method. In fact, the integrand and the path of integration can be specified by the Laplace transform. Consequently, the latter solution is more methodical than the former one.”

“Applications of analytic continuation to the Weber–Schafheitlin integral (the right timing for a statement’s appearance): Suppose we choose the weakest possible conditions required in an argument to be our theorem’s hypothesis. If the argument has used the method of analytic continuation no more than once, then no confusion will occur. However, what should we do if the argument has used the method of analytic continuation more than once?

Proposing a new condition without collecting enough evidence in advance has a problem with the timing for its appearance. Therefore, whenever we use the method of analytic continuation, we should check and record if the change of the condition is needed so that we may easily clarify the relationship between cause and effect in the proof structure.”

“Integration on a Riemann surface with branch points: If we reduce a contour integral on a Riemann surface to an integral along a line segment, the value of the latter integral may depend on which sheet the line segment is in, while the former integral is an invariant quantity. When we reduce a contour integral on a Riemann surface to an integral along a line segment, we often have to degenerate a part of the contour to a point. In order to make the argument of points along the contour continuous and simplify the calculation of these arguments, we should restore the degenerated point to its corresponding nondegeneate part.”

“Contour integrals for Bessel functions”

“The recurrence formulas for Neumann’s polynomials given by Watson [109, p.274, (1), (2) & (3)] can
be derived from
(see Watson [109] p.275, 1.9–1.10).

Remark 1. Want to prove uniform convergence when convergence is given
Remark 2. Series rearrangement
Remark 3. Detailed analysis.”

“Determine arg(1−t) on a contour around the branch point t = 1: We need a method rather than correct
results. Any step coming from guess may lead to the desired result this time; it may not next time. For
example, if the choice arg(−1) = π can lead to the desired result, we want to know why we cannot choose
arg(−1) = −π. Thus, if one provides correct results without a method, one may still make mistakes
sometimes. Ten correct examples are not as good as one correct method. Only when a complete method
is provided may we check if results are correct. When encountering a situation where a confusion may
easily occur, we should deliberately clarify the confusion rather avoid discussing it.”

“Binomial series: The classical view emphasizes the choice of principal value and the consistency with
previous results. The modern view emphasizes whether \( \sum_{k=0}^{\infty} \binom{n}{k} t^k \) is convergent and whether the cases
considered are inclusive.”

“Listing examples cannot be considered a proof: Listing examples cannot be considered a proof just like
a tangled ball of yarn cannot be called a piece of cloth. A professional proof must give the direction of
thoughts and the key idea. We should not avoid discussing the part difficult to describe. On the contrary,
we should work harder to give it a clear explanation.

A finite series must have the first term, the last term, and the general term. An infinite series must have
the first term and the general term. To figure out the general term of a series from its first few terms is
an example of inductive reasoning or a conjecture, but should not be considered a proof. For a binomial
coefficient, we should use its compact symbol \( \binom{n}{k} \) rather than its awkward factorial form \( \frac{n!}{k!(n-k)!} \) unless for
the purpose of computation. The formulas given in Hobson [52] p.106, 1.−8−.−1−] look messy due to the
abuse of notation. Suppose \( n \) is even. Hobson [52] p.107, (7)] expresses \( \cos n\theta \) as a finite series in ascending
powers of sine without the highest power term. Hobson [52] p.105, (3)] expresses \( (-1)^{n/2} \cos n\theta \)
as a finite series in descending powers of sine without the lowest power term. Hobson [52] p.107, 1.12]
claims that Hobson [52] p.107, (7)] is Hobson [52] p.105, (3)], written in reverse order. How is it possible
to compare two things when one of them is unknown?

Hobson [52] §78 & §79] expresses \( \cos n\theta \) and \( \sin n\theta \) as descending power series of sine. Their combi-
natorial proofs are tedious and annoying. If we want to express them in ascending power series of sine,
all have to do is list all the terms of the descending power series and then reverse the order. However,
Hobson [52] §80–§83] fails to do this simple way by repeating the same kind of tedious and annoying
combinatorial proofs. Mathematics is not for killing time. We have more important things to do.
Clarification of a point of confusion

“The proof of a theorem is hidden in the application whose proof requires the use of the theorem: Example.
Stirling’s theorem”

“Finding the inverse function of a given analytic function with the Fourier series method”

“Statements of a certain type have the same proof pattern”

“The Taylor series vs. the L’hôpital rule in terms of convergence: Sometimes, only after studying advanced
mathematics may we understand how we should properly deal with elementary mathematics. In
order to study infinite products of analytic functions, we must master the concept of uniform convergence.
Thus, it is important to see how the Taylor series and the L’hôpital rule affect convergence. Among proofs
for the case of point convergence, we should select the ones applicable to the case of uniform convergence.”

“Differentiation of a rational function whose denominator is a high power of a polynomial: One may use
the product rule \( (fg)' = f'g + fg' \) or use the quotient rule \( \left( \frac{f}{g} \right)' = \frac{g'f - fg'}{g^2} \). If one uses the latter rule, one
should not expand the two terms in the numerator of the resulting rational function. Cancel the common
factor of the numerator and the denominator of the resulting rational function first. This may avoid a lot
of unnecessary computations. If one were to expand the terms in the numerator after using the quotient
rule, this would make it difficult to cancel the common factor or identify the complicated resulting ex-
pression with the desired value.”
“The motive of creation and process of evolution for the method of steepest descents”
“With vs without guess and check: Proofs are used to check the truth of a statement and are not neces-
sarily helpful to understand its meaning. For example, we can use the mathematical induction to prove
\[ \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6, \]
but do not know how we get this formula. The proof is independent of the theme of this formula just like a quality control inspector checks only the packaging of product. This is a proof with guess and check; its analysis for the formula is shallow. We make the conclusion without enough confidence beforehand, and have to check afterwards; the guess and poor explanation lowers the quality of theory. Therefore, ideal and mature mathematical theories should gradually eliminate the guesswork in it. The features of a proof without guess and check: having a specific viewpoint; starting
with a careful plan to get the answer; all the operations being in control beforehand.”
“Infinite integrals: Tests of convergence: the comparison test, Abel’s test, and Dirichlet’s test. Tests
of uniform convergence: the method of change of variable, Abel’s test, Dirichlet’s test, and Lebesgue’s
dominated convergence theorem. Many theorems about uniform convergence can be considered corollar-
ies of Lebesgue’s dominated convergence theorem. We often evaluate infinite integrals by using Taylor
series expansions.
The process of evolution for Abel’s test for uniform convergence vs that for Weierstrass’ test”
“A science book author should not use definitions to stop readers’ questions: For any science book, a
reader should not accept a definition as a command about whose origin one should not question although
it does not require a proof. An author should not give a definition without providing a reason.”
“The right timing for correcting mistakes: In physics, we study facts. Theories are nothing but tools to
explain facts. When a theory fails to explain facts, it should be abandoned and eliminated. When we find
a statement contradictory to facts, we should trace to the origin of mistake and rewrite the theory from
there. Of course, an incorrect statement will lead to a lot of junk, but we are not interested in why they
are junk. The important thing is to correct mistakes as soon as they occur. Perhaps the Gibbs paradox
is valuable for books about the development history of statistical mechanics, but not for a textbook. A
textbook should not contain any incorrect statement because it is a reference book for quotation and ap-
lication.”
“Maxwell made a contradiction compatible by changing \( \nabla \times H = J_f \) to \( \nabla \times H = J_f + J_d \): the contradic-
tion to be resolved, his analysis, his remedy for compatibility, how the correction of the formula affects
the results whose validity depends on the formula, and other evidence of the existence of displacement
current.”
“Faraday made a contradiction compatible by changing \( \nabla \times E = 0 \) [static] to \( \nabla \times E = -\frac{\partial B}{\partial t} \) [nonstatic].”
“How we should properly treat Ampère’s law: the situation, our strategy, and the value of Ampère’s
law.”
“A more delicate and effective method provides more information.”
−10−1.−2] at best provides the idea of proof instead of a detailed proof. In order to highlight the key idea
and provide a rigorous proof, we should simplify our model and make it typical so that we may easily
generalize this special case to the general case. In other words, the following factors must be simplified:
the shape of \( C' \), the positions of \( P \) and \( P + ds \), and the solid angles.”
“Somewhat indirect calculations vs. direct calculations”
“The central-force method vs. the Coriolis method:
A. (Find the horizontal deflection by the plumb line caused by the Coriolis force acting on a particle
falling freely from a height)
I. (The central-force method) This method applies the entire formalism (General theory of central-force
motion) to a specific problem:
a. Derivation of the equation of motion (ellipse):
b. Analysis of the ellipse:
c. Express $T$ in terms of $\theta_0$ and compare the amount that the particle is deflected eastward with the amount that the point on Earth directly beneath the initial position of the particle moves eastward at time $T$.

II. (The Coriolis method) This method is tailored to the problem’s needs by following strategy:
   a. Separate useful information from unnecessary one.
   b. Treat the fictitious centrifugal force and Coriolis force as real forces.
   c. This method uses only Marion–Thornton [70, p.397, (10.34)]. We need not consider any concept given in I.a.

B. Euler’s method of tailoring a solution to the problem’s needs: He first assumes the solution (the fixed point $O$ on the sphere by the given rotation) exists. Then he finds its consequential property (the great circle $OA$ bisects $\angle AAa$). This observation allows him to conclude that a fixed point by rotation must have this property. Therefore, he first bisects $\angle AAa$ with a great circle. Then he attempts to look for the fixed point along the circle. Thus, he reduces his search scope from the entire sphere to a particle circle.

Conclusion: In most cases, we should concentrate on a small area and tailor a solution to problem’s needs when we try to discover or prove a unproven statement, find the origin for its discovery or the incentive for its proof, or look for the key idea behind the proof. In contrast, we should use the general theory to prove a specific theorem if the theorem is already proven and we want to see what role the theorem plays in the general theory from hindsight. In addition, if we divide the general theory into several categories, we would like to see to what category the theorem belongs for classification.

C. Infinitesimal rotations"

“Formal methods vs. heuristic methods: Lagrange’s equation of motion does not have a definite physical meaning because it can refer to $F = ma$ or $dM/dt = K$. When we apply it to a practical problem, we simply substitute the data into the equation without considering its derivation. Thus, a formal method puts physical meanings in a black box and plays with mathematical formulas alone. A formal proof makes it difficult for us to see the motivation behind it. Although the Lagrangian formalism is ineffective in local view, it is useful in global view. For example, it provides the mathematical foundation of the analogy between the two columns at the bottom of Symon [99, p.211]. Thus, it unifies the theory of rectilinear motion and the theory of rotation about a fixed axis. If we know the former theory alone, we have to use the analogy as a guide to study the latter theory.”

“A theory that leads to a contradiction can still be useful.”

“Classical derivation of the macroscopic Maxwell equations: Choudhury [20, §7.2] derives the macroscopic Maxwell equations from the microscopic Maxwell equations. The proof uses test functions rather than probabilities in quantum mechanics to define the concept of average. Thus, the approach considers quantum mechanics as a black box and fails to accurately indicate how the inner structure of quantum mechanics works in this case. Let us establish a closer relationship between the derivation and quantum mechanics.”

“Energies and forces under various conditions: vector analysis is the most effective method of determining the direction of force.”

“The orbit of a charged particle moving across a uniform magnetic induction is a circle. The proof given here reveals more insight with fewer calculations than that given in Wansness [106, p.534, 1.1–p.535, 1.10].”

“Solutions of Maxwell’s equations”

“Boundary value problem for a vector potential: Let $A_1 = A_1z\hat{z}$ and $A_2 = A_2z\hat{z}$. If $A_{1z} = A_{2z}$ on a $A_z$-equipotential line $C$, then $\nabla \times A_1 = \nabla \times A_2$.”

“Duality of electromagnetic fields”

“How we tailor calculations to our needs: We often do a lot of unnecessary calculations for the radiation zone: Sadiku [89, p.591, 1.1–l.1–1; p.595, 1.1–1.4; p.599, 1.1–1–p.600, l.–8] and Wansness [106, p.477, l.–5–p.478, l.3; p.482, l.–2–p.483, l.4]. However, most of them will never be used. p.734, l.–11–l.–2 in http://www.ece.rutgers.edu/~orfanidi/ewa/ch15.pdf shows how we should tailor calculations to our needs by avoiding unnecessary ones. The unnecessary calculations not only waste time and space but may also easily leave a gap in the theory due to the failure to provide the calculations that we should.”
“A method had better emphasize its key ideas rather than the general outlook of the final result: A method had better emphasize its essential ideas rather than the general outlook of the final result. By doing so, the description of the method will follow the natural thought flow: the cause first, the effect next. If we emphasize the general outlook of the final result, then the description of method will go against natural thought flow. The latter approach chooses the hard way; one can hardly see the insight from it.”

“Only through a language tool that is accurate enough may a delicate statement be described: The proof of $\lim_{\delta x \to 0} \frac{1}{\delta x} (\int_{\Sigma} F dV' - \int_{\Sigma} F dV) = -\int_{\sigma} F \rho_d dS'$ [Born–Wolf [13, p.899, (4)]] is confusing in both notation and language. The language tool that the authors use is neither clear nor accurate enough to describe such a delicate result.

“Marcoscopic versus microscopic viewpoints: Snell’s law says that the time required for the incident wavefront passes through $AD$ equals the time required for the transmitted wavefront passes through $AD$ [Hecht [50, p.100, Figure 4.19]; Born–Wolf [13, p.38, (1)]]. In Wangsness [106, chap. 25], Wangsness [106, p.408, (25-18; p.416, (25-49)] are derived from macroscopic Maxwell’s equations, especially on the boundary conditions [Wangsness [106, p.406, l.12]]. Thus, Maxwell’s equations is the common root of Snell’s law and Fresnel formulas. These two theorems can also be derived from Ewald–Oseen extinction theorem [Born–Wolf [13, p.108, (23)]. Based on atomic theory, we can have a deeper understanding about the transmitted waves.”

“$\delta$ is a linear functional: That the authors fail to point out this important concept hidden behind this proof makes one doubt if they master distribution theory.”

“A scientific textbook should be written with the audience in mind: it should contains not only results, but also the method to obtain them. If the proof is long, we should divide it into several steps so that readers may check the work step by step.”

“Tracing to the orientation’s root from which all its derivatives come: For any mathematical concept, we should trace it to its origin so that we may understand it more deeply. Spivak [95] is burdened with manifolds and Kreyszig [61] is burdened with tensors. Manifolds are the generation of Euclidean space and tensors are convenient for coordinate changes, but they are not the essential kernel of differential geometry. In contrast, O’neill [74] and Weatherburn [110] are concrete and intuitive, so they are good for practical usage.

The definition of involute given in Kreyszig [61, p.52, l.18] is original. [http://mathworld.wolfram.com/Involute.html] uses the consequential property [Kreyszig [61, p.52, (15.2); l.4–l.1]; Weatherburn [110, vol. 1, p.30, l.1–l.4]] of the above definition as the definition. The former definition is a simple characteristic property, while the latter definition provides one procedure of construction. When considering the converse problem [Kreyszig [61, p.53, l.1–l.4]], we would like to choose the former definition because there are fewer steps required to be reversed. The latter definitions destories the symmetry between involutes and evolutes."

“Reading classics in modern times: To construct normals at consecutive points involves two limiting processes: the construction of normals is one and the construction of a circle tangent to the curve is another one. Which one should be done first? If we exchange their order, can the results be different? The argument given in Weatherburn [110, vol. 1, p.66, l.14–l.15] fail to answer any of these questions. Therefore, if we want to preserve the original idea, we should complete one of the limiting processes first. Kreyszig [61] can be considered a bridge between the classic textbook Weatherburn [110] and the modern textbook O’neill [74]. Classic textbook pays too much attention to the computations on matrix elements of an operator, while the modern textbook tries to attach clear geometric meanings to the operator. For example, the definition given in O’neill [74, p.196, Definition 2.2] is compatible with the second formula of O’neill [74, p.58, Theorem 3.2]. Note that Kreyszig [61, p.83, Theorem 24.2] is not as clear as Weatherburn [110, vol. 1, p.62, (22)]. How Struik [98] improves classical differential geometry and how formalism invades modern differential geometry: Formalism may be convenient for application and easily make an argument rigorous, but may hide true geometric meanings and lose the natural and original taste.

The drawbacks of classical language: A concise definition cannot be not easily isolated from a long context; the background information is unclear and confusing; proofs are not rigorous.
Modern mathematics lacks depth and completeness. Actions speak louder than words: improvement by applying differentiation to linear algebra. In modern differential geometry, the form of differential equation for curves has a closer relationship to the parametric representation of these curves. Modern differential geometry is more organized and reveals more insights and geometric meanings. The modern version is not as clear, heuristic, complete, and organized as the original version and may easily emphasize on the trivial part.”

“How we name a definition: We could give asymptotic directions a more direct name such as self-conjugate directions, but for the mathematicians in the nineteen century the original picture was their first choice for naming a definition.”

“Advantageous viewpoints: When studying mathematics, we should take an advantageous viewpoint to get to the heart of the matter in few words. Weatherburn [110, vol. 1, p.106, l.−15–l.−7] says a lot, but it fails to hit the heart of the matter.”

“We should look for the clue to a solution in concrete examples: Theory is the framework of mathematics, while examples are the flesh of mathematics. When we try to solve a theoretical problem, we should resort to concrete examples for clues.

Natural order of thought flow: From the hindsight, we see that the following definition is more natural and original than the definition given in Kreyszig [61, p.48, Definition 14.2]:

Let \( G(\alpha_1(s), \alpha_2(s), \alpha_3(s)) = \sum_{n=1}^{\infty} a_n s^n \). The surface \( G(x_1, x_2, x_3) = 0 \) has contact of order \( m \) with the curve \( \alpha \) iff \( [a_i = 0 (l = 0, \cdots, m) \) and \( a_{m+1} \neq 0] \).

In fact, this is the only natural way to define contact of order \( m \). Thus, we should have used this new definition and Kreyszig [61, pp. 48–49, Lemma 14.2] to prove Kreyszig [61, p.48, Definition 14.2] as a theorem.”

“Reasons that lead to confusion or difficulty: The statements fail to manage complexity with simplicity, provide the details, hit the heart of the matter, focus on the key idea, use the right terms, distinguish one standpoint from the other, or indicate the conditions under which a statement is true.”

“The prototype of a concept and the compatibility between it and its derivatives: A definition of a mathematical concept contains a starting point (i.e., standpoint or viewpoint) and a special path leading to the concept. We should put the starting point at the prototype of the concept rather than its equivalent derivatives because of the advantage of viewing all the paths leading to the concept. Whenever we read a derivative for the first time, we should check the compatibility between it and the prototype. If in a textbook the author puts the prototype after its derivatives, the readers will actually lose the opportunity of organizing the topic by checking the compatibility of a derivative with the prototype.”

“The consistency of derivatives; the theory of differentiation in tensors runs parallel to that in differential forms; the consistency between the derivative of a map from one surface to another surface and the derivative of a map from one Euclidean space to another Euclidean space.”

“Name justification for elliptic, parabolic, and hyperbolic points”

“Motivation for proving a theorem about tensors: I. By a linear algebra approach, we may treat a symmetric tensor as a symmetric linear transformation \( B \). This approach inspires the general direction that we should take but not details. This is because the coordinate change formula for tensors and the coordinate change formula for linear transformations look different. II. A differential geometry approach may inspires the details because the coordinate change formula for tensors and the coordinate change formula for shape operators coincide. It is heuristic and insightful to connect symmetric tensors with symmetric matrices and shape operators.”

“Advantages of tensors: 1. Tensors condense information. 2. Avoid redundancy. 3. The information in tensor form is well-organized so that we may handle complicated cases easily. 4. Tensors and their local coordinates are two sides of one body. If Carmo [18] were to use tensors, Carmo [18] §3.2 & §3.3 could be incorporated into one. 5. Tensor notation makes it easier to compute, to trace origins and to link concepts, so tensor is a good tool to keep the description of things clear, concise, and complete, especially in complicated situations. By using tensor notations, it takes only three pages to prove Bonnet’s fundamental theorem of surface theory.”

“Interpretations from two different aspects of the same entity”
“From the Frenet formula to the connection equations (How to fish geometric information—the intrinsic property of geometric information): Based on the experience of studying curves, Frenet finds that geometric information often comes as a set if we adopt a frame wisely. Without using the entire set of frame, it would become difficult for us to clarify the meaning of a geometric object (e.g. Definition of geodesic curvature given in Weatherburn [110, p.109, l.9]), to organize collected information, or to get the big picture. If we use the Frenet frame as a fishing net, we can easily obtain all the useful information about a curve. A point on a curve has only one direction, while a point on a surface has many directions. To correct this problem, Élie Cartan designs another fishing net for surface [connection equations; O’neill [74, p.248, l.9]]. Only through connection equations may the discussion on various curvatures and the torsion be clear, teamlike and complete [O’neill [74, pp.230–231, Exercise 7; p.250, Exercise 1]]. The connection equations should be treated as a useful fishing tool for geometric information rather than a warehouse of information.”

“Motivation vs. verification: If a problem is given and we do not know what the answer is in advance, then the motivation to figure out the answer is required. If the answer is given [O’neill [74, p.254, l.–13–l.–12]], then we just need a verification [O’neill [74, p.254, l.–11–l.–10]]. However, for a mathematician, asking from where the answer comes is always more interesting than simply verifying the answer.”

“A proof should be natural and straightforward: one should not make a great fuss about little things.”

“Take one thing at a time”

“Definition’s accessibility and effectiveness (Differentiable functions on a regular surface): For application, the conditions given in a definition should be simple and there should be an easy and effective way to check the definition. A definition does not require a proof, but this does not mean that a definition can be used as a black-box-warehouse for piling up unsolvable problems even though they may be solvable in special cases. In the domain that we have no way to exercise our judgment, or for the verifying procedure that requires infinite steps, how can we tell whether a statement is true or not? If a definition is wrapped in multi-levels of black boxes, I do not think it could conjure up any useful images.

O’neill [74] p.124, l.3–l.11; p.182, l.4–l.18; p.184, l.5–l.15] discuss the goal, motivation, and the method of generalization in constructing surfaces or differentiable manifolds. For the definition of abstract surface, we ignore the logical impossibility [O’neill [74] p.182, l.17–l.18]] of effective testing the definition and try to treat the conditions as an axiom for the structure of surface [O’neill [74] p.182, l.18]]. For raising effectiveness, Carmo [18] p.70, Proposition 1] can be proved as a theorem instead of being treated as an axiom [O’neill [74] p.182, Definition 8.1(2)] using O’neill [74, p.125, Definition 1.2]. In the beginning stage of defining a $C^\infty$-manifold, Spivak introduces conditions one at a time based on needs [Spivak [95] vol. 1, p.35, l.10; l.–1]). However, as he introduces atlas and the maximal atlas, the theory’s quality deteriorates because some unnecessary considerations [Spivak [95] vol. 1, p.37, l.2–l.4; p.37, l.–8–p.38, l.4]] are thrown into the definition. His excuse is saving words [Spivak [95] vol. 1, p.37, l.–1]]. In fact, his formulation is simple in words, but complicated in thoughts, while O’neill [74] p.184, Definition 8.4] is simple in thoughts because it is a lean and mean generalization from O’neill [74, p.125, Definition 1.2].”

“Theory vs. application: In the study of mathematics, theory tends to deal with the difficult parts, especially the hidden one; while user-friendly applications tend to avoid dealing with the difficult parts. From the viewpoint of application, theorems in a theory are like packages or software products designed for completing a mission in an easy way. A user-friendly software product absorbs all difficult parts so that users need not touch them. To get the result, all one has to do is put data into it. If one tries to prove a statement by definition instead of applicable theorems, there would be lots of odds and ends to deal with and one would inevitably bump into the difficulties in reconstructing the theory. Sometimes a theorem has many versions: Spivak [95] vol. 1, p.41, (3)] & Blaga [11, p.120, Lemma]. They have different functions. In application, the version given in Spivak [95] vol. 1, p.41, (3)] is used most often. In contrast, the version given in Blaga [11] p.120, Lemma] points out directly the key to the proof of the theorem.”

“In order to keep up with the modern research, we should adopt a new viewpoint toward the inverse function theorem: The inverse function theorem usually refers to the version given in O’neill [74, p.161,
Theorem 5.4. In order to keep up with the modern research, we should adopt a new viewpoint toward the inverse function theorem and interpret it as the following natural and complete version that can be illustrated by a geometric figure (i.e., a linear isomorphism between tangent spaces is equivalent to a diffeomorphism between coordinate neighborhoods)."

"How a mathematical passage should be formulated: Writing a mathematical passage should not be like opening Pandora’s box or listing a bunch of statements whose truth needs to be validated. A passage should

(1). have a central concept, a central theme, and a key statement,
(2). take an advantageous viewpoint that may broaden our vision or make us see clearly the role played by each individual in the overall situation,
(3). be structured in levels; more precisely, be proceeded from the central theme outward level by level, and
(4). be organized in a systematic way so that the entire passage circles around the central theme. Otherwise, a disorganized passage only shows that its author fails to master that topic.

The environment is like a dark room and pointing out the central theme is like turning its lights on."

"A definition should avoid using abstract concepts and strange symbols and use concrete concepts and traditional methods: If the formulation of a definition is consistent with a known theory, we may just quote the theory rather than rebuild it. Avoiding the use of formalism may help see the insight behind the formulation. An abstract concept is often obtained by cutting a piece from the whole, breaking its outside links, weakening its effectiveness, emptying its contents, and considering it as a closed system of its own. The gain of black-box mechanism is at the cost of insights, motivations, and the big picture. A definition obtained by assembling a series of black-box mechanisms will not help visualize its geometric image."

"Conjugate directions at an elliptic or hyperbolic point of a surface: I. The origin: conjugate diameters in \(\mathbb{R}^3\) in terms of diametrical planes; conjugate diameters in \(\mathbb{R}^2\) in terms of diametrical lines. II. The characterization of conjugate directions in each of the following aspects: the second fundamental form, the self-adjointness of the differential of the Gauss map, the Dupin indicatrix, conjugate diameters of the Dupin indicatrix, and developable surfaces having contact with a surface \(S\) along a curve on \(S\)."

"A definition of a concept should directly explain what it is in simple words first: If its origins and evolution history are introduced first, the readers may ask if the origins and evolution history are the indispensible parts of the definition or if it is not possible to obtain the concept by other methods."

"Mathematical training is to teach students the standard methods of removing obstacles: A textbook in mathematics should be carefully written. The shortcomings of a popular one could have affected the later textbooks for centuries. A case in point is the topic on principal directions at a point on a surface. The essence of topic lies in its geometric meanings rather than computation details. Differential geometry contains a lot of topics, so the mastery of differential geometry requires a high level of organizing skills."

"The case classification for Bertrand curves: Suppose a regular curve \(C\) with \((\kappa, \tau)\) is given. We want to find its conjugates \(C_1\) with \((a, \alpha)\). The curve and conjugates are related by the formula given in Weatherburn [110, vol. 1, p.35, l.9], where \(\kappa, \tau\) are functions; \(a, \alpha\) are constants. Let us call \(\kappa, \tau, a, \alpha\) parameters. If the condition of one parameter is specified, it could affect other parameter values. This property makes it difficult to determine when we have completed the discussion of a case concerning a certain parameter or what remains to be done in the process of case classification."

"Intuition vs. rigor: the Möbius strip is not orientable."

"Bonnet’s fundamental theorem of surface theory:

I. This theorem lays the foundation of differential geometry. Its proof uses the theory of partial differential equations. It is interesting to see that PDE theory has such a useful geometric application.

II. The discussion here is based on Blaga [11, §4.17.4].

III. Tensor notation makes it easier to compute, to trace origins and to link concepts, so tensor is a good tool to keep the description of things clear, concise, and complete, especially in complicated situations.

IV. 1. The kernel and PDE component of Bonnet’s theorem is the following theorem:

PDE Theorem. Let Levi-Civita [67] p.14, (4′)] be a system of PDEs with initial conditions. If Levi-Civita
can be proved as local identities using the relations given in Levi-Civita \([67], p.14, (4')\) alone, then there exists locally a unique set of solutions for the system.

Remark 1. The difficult part lies in how to interpret Levi-Civita \([67], p.15, (5)\).

Remark 2. My criticism on the analysis given in Levi-Civita \([67], §II.2; §II.3\).

V. Struik \([98], p.135, Exercise 21\]

The proof requires a lot of computations. To prevent readers from getting lost, I will divide the proof into several stages so that for each stage readers have a small goal to achieve and a small result to check if they have made any mistakes in this stage.

VI. The idea of reducing PDEs to ODEs may greatly simplify the above proof.

“To be reader-friendly, a textbook should use common notations and present its proofs concisely: If a textbook contains strange notations, readers may have to search for the entire book to find where they appear for the first time unless it has a notation index. Thus, readers have to waste a lot of time just for finding the meaning of a notation. Copyright may lead an author to choosing different proofs or notations. In my opinion, it is appropriate to apply copyright to the discussion or interpretation of formulas, but not directly to formulas themselves.

It takes only fifteen lines to prove Kreyszig \([61], p.139, Theorem 42.1\) by using unit-speed curves, while it takes almost two pages \([Blaga [11], p.178, l.13–p.179, l.1–l.9]\) to prove the same theorem by using arbitrary-speed curves. There are no new ideas in the latter proof. The use of arbitrary-speed curves in the latter proof complicates each step of the former proof. Thus, in proving a curve theorem, we should parameterize a curve by its arc length \(s\) rather than arbitrary \(t\). The chain rule can easily convert the version of unit-speed curve to the version of arbitrary-speed curve.”

“The variation of the main body vs. the variation of its accessories: The Frenet approximation may help us predict theorems about the shape of a curve. In contrast, the Taylor series approximation has no way to provide any information about the shape of the curve.”

“Contact of finite order: I correct the mistake in Kreyszig \([61], p.49, l.9–l.8\) and fill the gap in Struik \([98], p.23, l.10–l.9\]. My discussion on contact of finite order will circle around these two themes.”

“The Lie–Darboux method of resolving an illusive contradiction by substituting true solutions into the required conditions: Struik \([98], p.37, l.4–p.38, l.19\) is at best a simple, efficient tool for testing solution candidates and provides crude information for our decision about selection. Its advantage lies in the fact that complicated choices can be removed in advance. However, we have to impose some conditions like \(\beta_0 = \beta_0 = 0\) and \(\alpha \alpha = \alpha \alpha = 0\). These restrictions might be the reason that leads to contradictory conclusions: \(c_3 d_3 = 1\) and \(c_3 d_3 = 1\). It may be possible to obtain the choices given in Struik \([98], p.38, l.16\] by other means without imposing the above restrictions. Once the restrictions are removed, the above contradiction may disappear. Consequently, until \((\alpha, \beta, \gamma)\) are definitely determined, it is simply not a proper timing to discuss compatibility. This is the first mistake that Struik commits. We may also treat this problem from the viewpoint of calculation. Although the calculation in checking if the given \((\epsilon, d)_i\)’s \([Struik [98], p.38, l.16]\) satisfy the equations in Struik \([98], p.38, l.10–l.13\) does not involve \(f\), we still have to face the difficulty of calculating \(\alpha \alpha = \alpha \alpha = 0\). Thus, at this stage the operations involving \(\alpha \alpha = \alpha \alpha = 0\) fail to have definite value. It is not the mature time to discuss compatibility. Therefore, it is better to find \((\alpha, \beta, \gamma)\) first and then check if they satisfy the equations given in Struik \([98], p.37, l.6\]. Since this calculation involves only definite values, the compatibility problem will not occur. Struik fails to complete this type of verification. This is the second mistake that he commits.”

“Developables: I. Situation control: Intrinsically, theorems about envelopes are local theorems. II. Making the concept of consecutiveness rigorous by using Rolle’s theorem. III. The duality between space curves and developables does not mean much.”

“The remarkable theorem considered by Gauss: The Gauss equations given in Weatherburn \([110], vol. 1, §41, (1), (2), (3), (4)\] naturally leads to the remarkable theorem considered by Gauss \([Weatherburn [110], vol. 1, p.93, (5)]\). Consequently, its proof given in Weatherburn \([110], vol. 1, p.93, l.14–l.7\] is natural and insightful.”

“The key to the proof of the equivalence of Codazzi equations and the compatibility conditions for Weingarten equations lies in the insight into inner structures rather than long calculations.”

“Following intuition is the best proof method: A proof using unnecessary objects may obscure the theo-
rem’s theme and confuse readers.”

“Direct and intuitive definitions of differential of a function on a surface: By intuition (straight line → curve), we may immediately obtain the direct and natural extension: O’Neill [74, p.11, Definition 3.1 → p.149, Definition 3.10]. The naming of covariant derivative of a geometric surface is symbolic in a sense.”

“Parallel postulate: Before Riemann, there had been many mathematicians who attempted to deduce the parallel postulate in \( E^2 \) but to no avail. Let us see how Riemann deals with this problem. We pay special attention to where he looks for counterexamples and how he obtains the answer. Both [pp.83–86, D. Hilbert, the Foundations of Geometry, La Salle, IL: the Open Court Publishing Company, 1950] and https://en.wikipedia.org/wiki/Parallel_postulate fail to grab the key idea of Riemann’s solution.”

“How we round off a corner of a curve”

“An incorrect definition leads to an incorrect proof: When an author fails to make readers understand his or her proof of a theorem, it either means that the proof is incorrect or means that the author fails to grasp the key idea of the proof. Such a “proof” wastes not only the author’s time, but also the readers’ time. Some people prove the existence of maximal atlas using the axiom of choice. The existence in the axiom of choice is assumptive, so the existence of maximal atlas produced by such proof is also assumptive. The purpose of the theory of axiom of choice is to see what consequential results would be if we were to consider it true. Its advantage: If we can prove the validity of the axiom of choice for a special case, then all its consequential results will be true for that special case. However, before we prove that the axiom of choice for the special case, the above consequential results should not be treated as true theorems. In contrast, Arnold [4, p.292, Example 3; p.291, Fig. 237] proves the consistency of any two of three big charts. It shows how to remove the obstacles of the most impossible case for consistency. Once their consistency problem is solved, to solve the consistency problem for any other two charts would be similar and easier. Consequently, this existence of maximal atlas is constructive.

Each of the three atlases given in Arnold [4, pp. 291–292, §33.3] can represent the maximal atlas containing it because the domain of each of its charts cannot be extended further so that the most impossible cases for consistency among the charts in the maximal atlas would be the cases for determining if the charts in the given atlas are consistent.

The differentiable manifold structure is defined as an equivalent class of atlases [Arnold [4, p.290, Definition 5]] or the maximal atlas [Spivak [95, vol.1, p. 38, l.−11]]. The drawback of the former definition is that we have to find an effective algorithm to determine if two charts are consistent before we can determine if two atlases are equivalent. Thus, the former definition may easily make us forget to check the consistency problem. The latter definition may contain too many extra charts which are useless in differential geometry. In my opinion, the definition of differentiable manifold structure most appropriate for differential geometry is using the latter definition and identifying the maximal atlas with the atlases that can represent it. That is, we should ignore the differences among them, but keep the distinction between them and the rest of atlases in the equivalent class. In differential geometry, we should accept set theory flexibly; in other words, we should transform it to a tool useful in differential geometry. Furthermore, for the atlas that can represent the maximal atlas, we keep a minimum number of charts in it as long as they are good enough for practical use.

The proof of Lee [66, p.13, Proposition 1.17(a)] actually uses the axiom of choice.”

“Differentiable manifolds vs. locally compact Hausdorff spaces”

“The strong version of Sard’s theorem”

“The indirect solving method by studying the problem’s background first sheds more insight on why we solve the problem this way.”

“Natural viewpoints vs. unnatural viewpoints toward tangent bundles: Mathematical development tends to become simple and natural. We do not care how many contents a textbook provides, how difficult it is to read these contents, or how odd the viewpoint that the author adopts is compared with the standard one, but we do care about if the author adopts the natural viewpoints to discuss the topic because the natural viewpoints make it easy to see the big picture.”

“The physical meaning of ODEs from the global view vs. that from a local view: Physical meanings may
inject new blood and new life into an abstract theorem in ODEs. They give its argument flow a guiding
direction and concrete meanings. The physical meanings of Spivak [95, vol. 1, p.203, Theorem 5] are
more clear and explicit from the global view; if we consider a local view alone, all we can see is odds
and ends rather than the big picture. This is because a local view preserves only a small part of the global
features. Without this big picture in mind, the discussion of Spivak [95, vol. 1, p.203, Theorem 5] would
become merely a display of a mess of meaningless formulas. In fact, a local view would make features
loom as if they are both hidden and present, seem as if there are both something and nothing. Thus, this
would make us difficult to express them clearly and logically. If one tries, it might turn out to be a wasted
effort.

Rectification: Suppose the solutions of an ODE are known. We try to use diffeomorphisms to map the
orbits of phase flow or integral curves of the direction field into curves of simple shapes.

Separation of variables: Suppose the solutions of an ODE are unknown. By a proper choice of new
variables, we can use the method of separation of variables to solve the ODE.

Thus the above two concepts are totally different. Except for Arnold [4, chap. 1, §6.6], Arnold [4, chap.
1, §6] essentially discusses the rectification of integral curves for homogeneous or quasi-homogeneous
ODEs. Unfortunately, Arnold somehow mistakes rectification for separation of variables; see Arnold [4,
p.76, 1.14–1.17]. It is important that we should not consider Arnold [4, p.79, l.1–l.17, Theorem]
[resp. Arnold [4, p.83, l.1–l.6, Theorem]] the method of separation for homogeneous [resp. quasi-
homogeneous] ODEs because we should not use a theorem itself to prove the same theorem."

“A textbook author should not omit a proof simply because it takes a lot of trouble to write it down
clearly: The author should provide at least the key idea of the proof. What readers need is methods rather
than results. The omission of methods only leaves readers groping in the dark. Very frequently, a proof
looks easy, but when one writes it out step-by-step, it may be not. There are also times when one finds
problems that one may not foresee at first. Thus, the omission of a proof can easily hide errors.”

“The integrability theorem involves by step-by-step adding geometric meanings; as the level gets more
advanced, its geometric meanings gets more generalized: using a typical example as a guide; calculus →
PDEs → differential geometry.

Both Spivak [95, vol. 1, p.204, l.9–p.263, l.1] and Hicks [51, §9.1] discuss the Frobenius theorem.
However, the essence of this topic contains only the following three theorems:

Theorem B: The ⇐ part of Theorem 3.8 of https://syafiqjohar.files.wordpress.com/2018/12/frobenius-1.pdf.

Theorem C: Hicks [51, pp.126–127, Theorem].”

“Characteristic property of a quotient structure vs. construction methods of the quotient structure:
A. Lee [66, p.605, Theorem A.27 (a) & (b)] belongs to the general type given in Bourbaki [15, p.280,
1.15–1.26].

B. For a particular mathematcal structure like topology, we may have a more effective criterion to char-
acterize quotient topology: Pervin [80, p.153, 1.2–1.5].

C. In order to give Bourbaki [15, p.280, (Fl)] a natural look, we may have the following view:
The statement given in Bourbaki [15, p.280, l.18] classifies, organizes, and summarizes the information
given in Bourbaki [15, p.280, l.16].

Strictly speaking, Lee [66, p.309, Proposition 12.7] is a generalization [Lee [66, p.605, l.5]] rather than
an example of quotient structure because A is multilinear rather than linear. However, the underlying
idea of tensor product space and quotient topology is the same, so their theory developments are similar.
The characteristic property of tensor product space does not directly prescribe any construction method
of tensor product space, but the resulting tensor product space by any construction method cannot violate
the characteristic property.”

“Recovery of skills in definition design:
I. In an axiomatic system, we give axioms and definitions first, and then derive theorems from them.
Thus, in an axiomatic approach to developing a theory, we must have the foresight of making it con-
sistent with the existing theory when introducing a new defintion. The belief of its truth for readers is
supposed to form in the future. However, a definition is usually given without any explanation. Its legality relies on the rationale that you will not get a contradiction as you proceed. In order to put it on a more solid foundation, we should not blindly accept it. How can we predict its truth? How can we find clues for its justification? In other words, we should ask how the definition is designed. That is, we should recover the skills of definition design.

II. Skills of definition design: A definition should be natural and intuitive; the underlying universal principle of definition design should be revealed.

“Telling the nuances between an algebraic dual spaces and a Banach dual space to clarify confusion: Being confused means that there is something that one needs to learn. If one understands only the statement and the proof of a theorem, then one says that one understands the theorem. This is not quite true. To test one’s understanding, a second theorem with similar hypotheses and opposite conclusion should be brought in and let one tell the nuances between the two to explain why the two theorem do not contradict each other. If one does not know what to do, this reveals that one’s understanding is shallow. In other words, consistency and thorough understanding are important. p.1, l.13–l.22 in https://kconrad.math.uconn.edu/blurbs/linmultialg/dualspaceinfinite.pdf provides such a case: it compares an infinite-dimensional vector space \( V \) and \( l^2 \) and explains why \( \dim V < \dim V^{**} \) and \( l^2 \cong l^{2**} \) do not contradict each other.”

“Different stances may make discussion get stuck and leave questions unanswered:
I. Suppose \( f : \mathbb{R}^3 \to \mathbb{R} \) is \( C^\infty \). Then the notation \( D^f_xv \) may have the following two meanings:
(1). The first meaning: (the matrix \((D^f_j)_i(x)\) of the differential \( D^f_x \)) [Rudin [86, p.191, l.17–l.18]] × (the column vector \( v \)).
(2). The second meaning is given in O’neill [74, p.23, Definition 5.2].
II. The scenario of https://math.stackexchange.com/questions/1120430/derivative-of-bilinear-forms is as follows: Let Q be the one who proposes the question and A be the one who answers the question.
Q: The notation \( D^f_{(x,y)}(a,b) \) means the first meaning to me. Since you interpret it as the second meaning, you fail to answer my question.
A: According to O’neill [74, p.23, Definition 5.2], \( D^f_{(x,y)}(a,b) \) means the second meaning. Consequently, I completely answer your question.
Q’s view: A’s answer is unsatisfactory because he fails to prove that \( f \) is differentiable. A should have proved the differentiability of \( f \) to validate his original argument in https://math.stackexchange.com/questions/1120430/derivative-of-bilinear-forms.
A’s or someone else’s view: It is O’neill [74] that should be blamed because it fails to prove the equivalence of the notation’s two possible meanings.
The discussion has gotten stuck and the questions have been left unanswered ever since.”

“One may increase reading efficiency for a tool book by 88 times if one has a goal in mind: A wrench is useless until one uses it to repair a pipe leak. The theorems in a tool book do not have meanings; the meaning of a theorem appears only when one uses it. In my opinion, a tool book, like a tool room, should provide a tool’s location and properties (usage). It should not contain any exercise. This is because most methods in a tool book are stereotype and the original idea for these methods can only be found in a broader and more inspiring area. Thus, a tool book should provide at least the exact location of solutions if it contains any exercise. Someone may say exercises help one’s thinking. Well, there are a lot of better things to do than solving exercises in a tool book.

When I was a university student, it took me six months to read Bourbaki [15] part 1, chap. I, §1–§2]. Then I decided to read other easier topology textbooks like Pervin [80] and Dugundji [28] instead. Now I need to solve Lee [66, p.611, Exercise A.54]. I have found that the solution is given in Bourbaki [15] part 1, chap. I, §10, no. 1; no. 2]. If I read the entire content of Bourbaki [15] part 1, chap. I, §3, no.1–§10, no.2] aimlessly and indiscriminately, based on my past reading speed, it may take me at least 22 months to complete this task. It may not leave any impression in a little while. However, this time I just need to solve Lee [66, p.611, Exercise A.54], so I may avoid reading any theorem unrelated to this purpose. If I need to use a theorem, I can read only the small section containing that theorem. In this way, I solve the
exercise in a week. 1 week: 22 months = 1 : 88.”

**Keywords.** Productive methods, accessibility, functions, reduction, analogy, effectiveness, simplicity, directness, flexibility, fully utilizing resources, hitting multiple targets with one shot, structurization, insightfulness, essence, networks, compatibility, unifications, avoiding repetition and unnecessary complications, method of weakening hypothesis, physical methods, physical interpretations, physical proofs, Leibniz integral rule, Cauchy’s theorem, Cauchy’s integral formula, residue theorem, Runge’s theorem, Dirac delta function, Heaviside function, Green functions, Bessel functions, Riemann zeta function, Lipschitz conditions, boundary conditions, regular singular points, ratio test, root test, Frobenius method, inverse function theorem, Riemann–Lebesgue lemma, prime number theorem, Baire’s category theorem, Pragmen–Lindelöf theorem, Paley–Wiener theorem, functional analysis, splitting field, perfect field, Lagrange’s resolvents, Galois resolvents, recurrence relations, generating function, Lengendre’s equation, integral transforms, Laplace transform, separation of variables, tensor product, wedge product, contravariant vectors, covariant vectors, fundamental groups, covering space, topological group homomorphism, covering group, evolute, involutes, centers of curvature, envelope, normals, normal incidence, singularity of the second kind, boundedness, growth condition, rotation operators, limit-point case, real nondecreasing spectral functions, Hermitian nondecreasing spectral matrices, Sturm’s oscillation theorem, Prüfer substitution, Poincaré phase plane, variational derivative, isoperimetric problems, holonomic problems, non-holonomic problems, positive semidefinite matrix, canonical Euler equations, characteristic system of the Hamilton–Jacobi equation, Hamilton–Jacobi theory, method of characteristics, Galilean transformations, Lorentz transformations, Michelson–Morley experiment, Maxwell’s equations, muon decay, special relativity, proper time, relativistic kinetic energy, relativistic Lagrangian, center-of-mass coordinate system, coupled harmonic oscillators, normal modes, generalized coordinates, exact differential, separation of variables, ruled surface, developable surface, diretrix, argumented matrix, polar lines, polar planes, osculating plane, osculating circle, osculating sphere, the second law of thermodynamics, Pfaffian forms, random variable, characteristic function, distribution, inversion formula, convergence in probability, almost sure convergence, strong law of large numbers, central limit theorem, cluster point, Ritz method, Sturm–Liouville problems, direct methods, method of finite differences, method of Lagrange multipliers, equation of the vibrating membrane, vector triple product, testing functions, contour integrals, dummy variable, branch points, Euler transforms, Fuchsian type, Borel measures, hypergeometric, confluent, regular singularity, Riemann’s P-equation, Lebesgue dominated convergence theorem, methodical solutions, Weber–Schaafheitlin integral, Riemann surfaces, Neumann’s polynomials, Bessel coefficients, Zhu–Vandermonde’s identity, method of steepest descents, Weierstrass’ test, Abel’s test, Dirichlet’s test, Cauchy data, Cauchy–Kowalevski theorem, microstates, entropy, extensive property, Gibbs paradox, displacement current, induced current, homopolar generator, magnetostatics, idealized circuit, solid angle, quasi-static approximation, capacitance, inductance, reversible process, total reflection, radiation pressure, kinetic theory of gases, Ewald–Oseen extinction theorem

Mathematical training is to teach students the methods of removing obstacles, especially the standard ones that will be used again and again. The more methods one has learned, the higher one’s skill and the more chances one may do some creative works in mathematics.

Providing a proof without a method involves giving the final answer by intuition first and then justifying it from hindsight (see the proof of Munkres [72, p.322, Theorem 1.2]). We may distill a method from the above justification. Understanding the method will enable us to systematically proceed toward the solution by analyzing patterns and taking advantage of the circumstances. Munkres [73] p.327, l.7–l.10; l.11–l.14; p.328, l.14–p.329, l.6] are parts of the method. A method is useful for generalization (Munkres [73, p.329, Theorem 51.3]).
A method is the summary of essential ideas for solving a problem. The solution guided by a method is often concise, organized, and insightful (Edwards [30, p.4, l.−2–p.5, l.13]).

1 The approach from the microscopic viewpoint vs. that from the macroscopic viewpoint

From the microscopic point of view, solving a problem is equivalent to exploring possibilities. From the macroscopic point of view, solving a problem is equivalent to eliminating impossibilities. For example, when we try to factor cyclotomic integers into ideal prime divisors, we may

1) Use the approach given in Edwards [29, p.128, l.−3–p.129, l.21] to construct ideal prime divisors (explore the possibilities) or

2) Use Stewart–Tall [97, p.186, Theorem 10.1] to find the ideal prime factorization in the general case and then prove that the $e_1, \ldots, e_r$ given in Stewart–Tall [97, p.186, l.8] are all equal to 1 using van der Waerden [102, vol.1, p.120, l.4–l.19] (eliminating the impossibilities).

2 How we choose the most suitable setting for illustrating a method

(Partitions of unity Munkres [72, p.222, Theorem 5.1])

The construction of a partition of unity has wide applications: topology, real analysis (Rudin [88, p.41, Theorem 2.13]), and differential geometry (Spivak [95 vol. 1, p.69, Corollary 16]). In essence, the construction of a partition of unity is a topological method. In order to ensure the method’s wide application, the setting should be general. Dugundji [28, p.144, Proposition 3.2] and the diagram given in Dugundji [28, p.311] show that a locally compact, paracompact, or normal topological space meets the setting requirement. In order to expressively illustrate a method’s essence, the construction should be simple. The method given in Munkres [72, p.222, Theorem 5.1] is simpler than that given in Rudin [88, p.41, Theorem 2.13]. The formulation and proof of Urysohn’s lemma given in Dugundji [28, p.146, Theorem 4.1] is simpler than those given in Rudin [88, p.40, Proposition 2.12] and Spivak [95 vol. 1, p.44, Lemma 2]. Furthermore, choosing a finite partition of unity will free us from considering the nuisance given in Dugundji [28, p.170, Definition 4.1(1)]. Except for settings, the constructing methods in Munkres [72, p.222, Theorem 5.1], Dugundji [28, p.170, Theorem 4.2] and Spivak [95 vol. 1, p.68, Theorem 15] are the same. All the above considerations make normal spaces the best choice of a setting for illustrating “partitions of unity”.

If we discuss theorems or solutions of differential equations, the method can be divided into three stages: input, process, and output. For a theorem, the input is the hypothesis and the output is the conclusion. For solving a differential equation, the input is the problem and the output is the solution. The output is an essential tool for determining the quality of a method in this case. If we discuss theories, proofs, or definitions, it suffices to consider input and process because we are interested only in their method. The output stage can be ignored in this case. The input is the settings and the process is the formulations.

Suppose we compare Method A with Method B. If the input of Method B is more than or equal to that of Method A and if both the process and the output of Method B are better than those of Method A, then we say that Method B is more productive than Method A. The method given in the general case can be applied
to specific cases. However, specific cases contain more resources, there may be more effective methods available for specific cases. Our goal is to seek the most effective method in each case.

3 Productive methods

3.1 Strategies for improving the output

When we formulate a theorem, the conclusion should be as strong as possible. Specialization often divides the discussion of a topic into cases and reduces results to the simplest form for each case. When we seek a solution, the solution should be specific and precise. The solving plan should be executed thoroughly and perfectly; the solution should be expressed in closed form if possible.

Example 3.1.

Although the form given in Watson–Whittaker [108, p.365, 1.2–1.3] is good for generalization (Watson–Whittaker [108 p.368, 1–9–1.1]), it is not as effective as the forms given in Guo–Wang [46, p.351, (5) & (6)]. First, the former form has not been reduced to simple form for each case, so it is not good for direct application. Second, if a series terminates, we want to know how many terms it has, otherwise the answer is not complete.

Example 3.2. (The construction of Green’s functions in one dimension)

The definition of a Green’s function is given by Ince [54, p.254, l.1–l.19]. The uniqueness of Green’s function follows from Ince [54, p.254, l.12–l.19] (Coddington–Levinson [22, p.192, l.16–l.20]). Indeed, as a function of $x$, $G – \tilde{G}$ is of class $C^{n-1}$ because $G$ and $\tilde{G}$ have the same discontinuity at $x = \xi$. Although Green’s function is unique, there are many methods for its construction. Here are some examples: Ince [54, p.254, l.1–l.19–p.255, l.1–11], Coddington–Levinson [22, p.190, l.12–p.192, l.1] (let $l = 0$), Gerlach [38, Theorem 45.1]. The first example provides a solution which proposes a plan but fails to execute it for the treatment of both the differential equation and the boundary conditions. The second example provides a solution which finishes the treatment of both the differential equation, but only proposes a plan for dealing with the boundary conditions without finishing the plan. The third example provides a solution which finishes the treatment of both the differential equation and the boundary conditions. The more precise the form is, the stronger properties of Green’s function we may obtain from it (Compare Ince [54, p.254, l.19–l.20] with Theorem 45.1). The first example provides a solution which proposes a plan but fails to execute it for the treatment of both the differential equation and the boundary conditions. However, specific cases contain more resources, there may be more effective methods available for specific cases. Our goal is to seek the most effective method in each case.

(1) Corrections for the first example:

(a) “$P(G, H) = 0$ when $x = a$ and when $x = b$” given in Ince [54, p.256, l.7–l.8] should be replaced with “$P(G, H)^{(p)} = 0$ (Ince [54, p.213, l.19])”.

(b) $p_0[H^{d^{n-1}}G] - G^{d^{n-1}}H$ given in Ince [54, p.256, l.11–l.12] should be replaced with $p_0[H^{d^{n-1}}G]^{(p)} + (-1)^{n-1}G^{d^{n-1}}H]^{(p)}$.

(c) $p_0(\xi_1)H(\xi_1, \xi_2)\lim_{\frac{d^{n-1}G}{dx^{n-1}}}^{n-1}G_{\xi_1^{-\eps}}^{n-1} - p_0(\xi_2)G(\xi_2, \xi_1)\lim_{\frac{d^{n-1}H}{dx^{n-1}}}^{n-1}H_{\xi_2^{-\eps}}^{n-1} = 0$ should be replaced with $p_0(\xi_1)H(\xi_1, \xi_2)\lim_{\frac{d^{n-1}G}{dx^{n-1}}}^{n-1}G_{\xi_1^{-\eps}}^{n-1} + (-1)^{n-1}p_0(\xi_2)G(\xi_2, \xi_1)\lim_{\frac{d^{n-1}H}{dx^{n-1}}}^{n-1}H_{\xi_2^{-\eps}}^{n-1} = 0$.

(d) “$p_0(\xi_1)\lim_{\frac{d^{n-1}G}{dx^{n-1}}}^{n-1}G_{\xi_1^{-\eps}}^{n-1} = p_0(\xi_2)\lim_{\frac{d^{n-1}H}{dx^{n-1}}}^{n-1}H_{\xi_2^{-\eps}}^{n-1} = 1$” should be replaced with “$p_0(\xi_1)\lim_{\frac{d^{n-1}G}{dx^{n-1}}}^{n-1}G_{\xi_1^{-\eps}}^{n-1} = (-1)^np_0(\xi_2)\lim_{\frac{d^{n-1}H}{dx^{n-1}}}^{n-1}H_{\xi_2^{-\eps}}^{n-1} = 1$.”
(2) Supplements of the second example: The proof of \( Lu = lu + f \) can be found in Ince \([54, p.256, l.6–p.257, l.10]\). The equality given in Ince \([54, p.257, l.1.5]\) follows from the Leibniz integral rule and 
\[
\frac{d}{dt} \left( \int_a^b f(x) \, dx \right) = \int_a^b \frac{df}{dx} \, dx + f(b) - f(a).
\]

(3) Supplements of the third example: Bernd \([8]\) provides the motivation of the construction of Green’s function given in Gerlach \([38, Theorem 45.1]\). Bernd \([8, Example 3]\) relates Green’s function to the Dirac delta function and the Heaviside function. Bernd \([8, (5.35)]\) motivates us to generalize the relationships to the abstract level of functional analysis (Rudin \([87, p.206, Exercise 10; p.378, l.6–6])). Compare Bernd \([8, (5.27)]\) with the formula given in Rudin \([87, p.206, l.9]\).

3.2 Strategies for improving the qualities of process

The qualities of process can be roughly divided into following categories:

(1) Accessibility: Construct the existence of solution in a finite number of steps. Avoid using any proposition whose validity cannot be verified in a finite number of steps. Avoid using the axiom of choice, reduction to absurdity, and mathematical induction. If we must use reduction to absurdity or mathematical induction, we should narrow its scope of application wherever possible. When using mathematical induction, we should reduce the amount of work in the induction step wherever possible lest the program takes too much time and memory in computer.

(Discussion) Some mathematicians think that existence can be established by construction, by the method of reduction to absurdity, or by assumption. Once the existence is established, we should not worry about the method of establishing the existence. However, mathematicians in the intuitionist school insist that only when every claim during the construction of existence can be determined to be true in a finite number of steps may the existence be considered mathematically significant.

A compound sentence is true only if each of its component sentences is true. If one of its component sentence cannot be determined to be true or false in a finite number of steps, then this compound sentence is mathematically meaningless. As for the choice of resources and tools, in principle, we choose only necessary ones (Edwards \([30, p.68, l.21–l.26]\)). Discard irrelevant and unnecessary ones (Edwards \([30, p.68, l.14–l.19]\)).

(2) Functions

(a) Reduction: Reduce calculations; reduce to lower-level systems. The canonical Euler equations represent the characteristic system associated with the Hamilton–Jacobi equation [Fomin–Gelfand \([35, p.90, l.12–l.10]\)]. Originally, this fact was a part of Hamilton–Jacobi theory in classical mechanics. Since then the method of characteristics has been developed to be an important tool in reducing a partial differential equation to a system of ordinary differential equations. Read Tkachev \([101, Method of characteristic strips]\) and Gibbon \([39, chaps. 1 & 2]\).

(b) Analogy: We should link a new concept with a familiar one so that we have a model in mind for studying the new concept. The method of analogical correspondences often proposes conjectures.

(Discussion) When we study a new concept, the first thing we should do is relate it to a familiar concept by establishing a major link between them. This is because analogy provides a vantage point to see the big picture. Before the link is established, every task is difficult. Once the link is established, every task becomes easy.
(c) Effectiveness: This quality refers to accessibility, specification, elementary methods, quantitative instead of qualitative formulations, the construction of solutions by an effective algorithm rather than trial and error, the use of simple theorems rather than complicated ones.

(Discussion) Although effective mathematics can use available resources to provide an effective method of constructing the strong existence of a mathematical object, it sometimes has congenital defects in other mathematical tasks. In general, if we try to emphasize efficiency, accuracy, concrete construction, an argument’s strength, utilizing resources, or other details, we move toward effectiveness. However, if we try to emphasize the whole, we move away from effectiveness. Such tasks are unification, classification, abstraction, generalization, clarifying structures (Wang [105, Example 5.12 (The Jordan canonical form)]), identifying the essential reason for uniqueness, or proving the result is independent of our choices of construction.

(d) Hit multiple targets with one shot

(e) Directness: Adopt a direct approach to the solution rather than a roundabout one.

(f) Simplicity: Reduce the general form to a simple form. If there are several methods available, we should choose the simplest one. Sometimes one method is always simpler than others; sometimes the choice for simplicity varies from case to case.

(g) Avoid unnecessary complications: Avoid repetition or awkward languages. We should not necessarily generalize a theorem unless the generalized theorem has practical applications. Our argument should use simple statements to prove complex ones, not vice versa. We should use theories as fewer as possible and choose theories as simple as possible. Only through removing unnecessary theories from our argument may we make our solving process leaner and simpler.

(Discussion) When we solve a problem, we should avoid using unnecessary theories. After Guo–Wang [46, pp.62–63] discusses the Frobenius method in the general case, Guo–Wang [46, §4.4] repeats the same method many times. According to the general theory of regular singular points (Guo–Wang [46, §2.4 & §2.5]), shall we repeat the same discussion for the Legendre equation’s three regular singular points given in Watson–Whittaker [108, p.304, l.18]? If we use a theory only because it is applicable, then our argument will become aimless. Fortunately, there are simpler methods available. Indeed, when we solve an ordinary differential equation, we should focus on directly finding a solution in closed form. We should not divert our attention to the solution’s properties or generalization. See Example 3.33.

(h) Advantages: Take full advantage of available resources, circumstances and opportunities.

(i) Flexibility: Use the ideas in a theory flexibly rather than the exact form of theorems by mastering the entire theory. That is, we should apply the essential idea rather than the exact form of the general theorem to a specific case.

(j) Avoid contradictions: A theory cannot allow contradiction or inconsistency. If a theory cannot explain certain phenomena, we should modify it so that the new theory can explain them and in special cases the results of the new theory should reduce to those of the old one.

(k) Expand the scope of application without loss of efficiency

(3) Structurization

Theorems would become fragmented without structure; without \( \sigma \)-algebra, the statement given in Borovkov [14, p.47, 1.8–1.9] would become fragmented [see Chung [21, p.65, Exercise 3]]. The critical structure may fail to emerge more often because of lacking in skilful analysis. In one variable, the derivative of a function at a point is a number; in several variables, the differential of a function at a point
is a matrix [Rudin [86] p.189, 1.–12]]. For the limit-point case at \( \infty \), we use a real nondecreasing spectral function [Coddington–Levinson [22] p.232, 1.5]]; for the limit-point case at both \(-\infty\) and \(\infty\), we use a Hermitian nondecreasing spectral matrix [Coddington–Levinson [22] p.247; 1.–17; 1.–10;1.–7]]. The basic idea of differentiation of one variable and that of several variables are essentially the same, so are one-end and two-end limit-point cases [Coddington–Levinson [22] chap. 9, §3–§5]]. In the complicated case, the only thing we should pay attention to is the formation of a new structure–matrix. The concept of self-adjointness and eigenvalues in matrix theory can be used to classify the systems of differential equations [Coddington–Levinson [22] p.189, 1.4] and find their eigenfunctions [Coddington–Levinson [22] p.196, 1.15–p.197, 1.8]]. Consequently, structurization may help us systematically operate and deeply understand the complicated cases.

(a) Insightfulness: This quality refers to origins, insights, motivations, perspectives, true nature, inner structures, and formal solutions.

(Discussion) A proof should be well-structured and insightful. If the conclusion of a theorem is valid in most cases, then we simply apply the conclusion to a problem without checking if the situation satisfies the hypothesis of the theorem. Thus, we use the theorem first and justify the application later. This formal procedure allows us to quickly obtain a solution candidate and have a crude blue print for solving the problem. In order to master the basic part of a subject, one should move ahead to study its advanced part.

(b) Essence: This quality refers to modeling, the common pattern of a solutions, key points, main veins, and the core of a theory. We should seek the common pattern of solutions in order to grasp the main vein that runs through the entire theory. Keen observations carry forward the method’s development. If we use this main vein as the guideline to develop the theory, it may help us organize our material and clarify the theory. See Example 3.51 and Example 3.52. In order to avoid confusion and complexity, we must reduce various solution strategies to the essence. We should structuring the problem and locate the first obstacle to the solution so that we may easily and quickly recognize the reason why the problem cannot be solved with the assigned tools. The frequently used statements in a theory should be considered valued basics. Studying a complicated theory without understanding its essence is like returning from a treasure mountain with empty hands. The essence often becomes clearer if we reduce a complicated case to a simple case. Only through reducing a method to its essence may we be able to easily deal with complicated problems.

(c) Flowcharts or networks: This quality refers to relationships, compatibility, unifications, interactions, interdisplines, integration studies, external links, links among milestones, the big picture, flow charts in design, proof strategies, and the evaluation or criticism of a theory. A theorem is inseparable from its role in the entire theory. Only through advanced researches may we correct our mistakes in basics. A math network strengthens effectiveness. See Example 3.70.

(Discussion) Gauss, Lagrange, Kummer, and Hilbert wrote important work after mastering mathematics and physics. Their deep understanding of mathematical network enabled them to write masterpieces. It would be difficult to do so for those who specialize only a narrow field. Galois was able to write great work because he had read Lagrange’s opus. Einstein had also read many people’s work before he wrote papers on relativity. Mastering mathematical networks may help us deduce simplicity from apparent complicity, recognize the essence, understand the situation, propose important questions, and write significant papers.

A modern approach to a topic often focuses its study on a local, isolated and self-contained system. This approach will make it difficult to see the topic’s origin and its role in the entire theory. Consequently, we should keep the external links open to help preserve the origin and the big picture.
The following examples indicate in parentheses the qualities of process for the methods under discussion.

**Example 3.3.** (Accessibility)
There are ghosts or no ghosts. This proposition is mathematically meaningless even though there are no other possibilities in logic.

**Example 3.4.** (Accessibility)
Suppose we want to prove the existence of the splitting field of a polynomial (Edwards [30, §51]). We must provide factorization methods which enable us to determine if a polynomial is reducible or irreducible in a finite number of steps (Edwards [30, p.69, 1.24–1.28; p.72, 1.15–1.19]). In order to avoid using unnecessary tools and resources, we consider only the following relevant statements:

1. A polynomial with integer coefficients is either reducible or irreducible. This statement was proved by Kronecker (Edwards [30, p.72, 1.21–p.73, 1.19]).
2. A polynomial with rational coefficients is either reducible or irreducible (Edwards [30, p.73, Corollary 1]).
3. Given a factorization method for the coefficient field $K$, one can find a factorization method for the coefficient field $K(a)$ obtained by adjoining to $K$ an indeterminate $a$ (Edwards [30, §§58–§59]).
   The key to proving the above statement is to reduce $f(a,x)$ with two variables to $\tilde{f}(t) = f(tN,t)$ with one variable (Edwards [30, p.76, 1.11–1.13]). This idea came also from Kronecker.
4. Given a factorization method for the coefficient field $K$, one can find a factorization method for the coefficient field $K(a)$ obtained by adjoining to $K$ a root $a$ of an irreducible polynomial with coefficients in $K$ (Edwards [30, §60]).
   The key to proving the above statement is to use the method of undetermined coefficients by considering the factorization of the norm $Nf(x+ua)$ (Edwards [30, p.78, 1.5 & 1.10]).

**Example 3.5.** (Accessibility)
(Baire’s category theorem) (Royden [85, p.139, Corollary 16]; Dugundji [28, p.251, Ex. 6])
The existence given in Dugundji [28, p.300, Theorem 4.2] is derived from reduction to absurdity. This existence is not as effective as the constructive existence given in Gelbaum–Olmsted [37, p.38, 1.6–p.39, 1.3]. The fact that modern mathematicians rashly adopt short proofs (Royden [85, p.141, Exercise 30.d]; Rudin [88, p.121, Exercise 14]) but neglect effectiveness will reduce the quality of theory. Munkres [72, §7–§8] tries to improve the effectiveness of the existence given in Dugundji [28, p.300, Theorem 4.2]. The attempt is futile because Munkres’ use of Baire’s category theorem has seriously ruined effectiveness in the first place.

**Example 3.6.** (Accessibility)
(Constructing continuous functions that are non-differentiable)
In each of the sections in Titchmarsh [100, §§11.21, §11.22 and §11.23], Titchmarsh constructs a continuous function that is not differentiable. The first one is simplest. This shows that we should start a project with a small task. The first and the third example show that if the derivative were to exist, it would have two different values. The second example shows that if the derivative were to exist, its value would be $+\infty$. Thus,
the three constructions and proofs are similar. The reduction to absurdity Titchmarsh uses can be considered trivial. Thus, Titchmarsh’s proofs are effective. In contrast, modern mathematicians love to use a non-trivial (see Example 3.5) reduction to absurdity to construct continuous functions that are non-differentiable. Due to their negligence the method of construction in modern textbooks deteriorates.

**Example 3.7.** (Accessibility; insightfulness; avoiding unnecessary complications)

(Cauchy’s theorem)

In order to prove a theorem effectively, the quoted theorems in its proof should be simple, practical, and indispensable. The proof given in Rudin [88, p.221, Theorem 10.13] is more effective than those of Rudin [88, p.235, Theorem 10.35] and Saks–Zygmund [90, p.177, Theorem 2.3]. Rudin [88, p.224, Theorem 10.17] quoted in Rudin [88, p.236, l.1–l.5; l.11–l.26] obscures the essence of Cauchy’s theorem. Rudin [88, p.221, Theorem 10.13] and Saks–Zygmund [90, p.177, Theorem 2.3] are more practical. For example, if the equality holds for $(i, j, k)$, then its holds for $(i, j, k, l)$. Thus, it suffices to consider the cases when $(i, j, k)$. Thus, it suffices to consider the following three cases: $(i, j) = (1, 2), (k, l) = (1, 3); (i, j) = (1, 2), (k, l) = (2, 3); (i, j) = (1, 3), (k, l) = (2, 3).

**Example 3.8.** (Accessibility; directness; simplicity)

Both van der Waerden [102, vol.1, p.124, 1.8–1.9] and Edwards [30, p.99, 1.7–1.8] define the concept of a perfect field. The later definition is more accessible than the former one.

**Example 3.9.** (Accessibility; avoiding unnecessary complications)

Briefly speaking, Galois theory contains only two theorems: Edwards [30, p.59, 1.7–1.11; p.61, l.12–l.8]. The former theorem builds only the Galois subgroup corresponding to an extension of the base field $K$ obtained by adjoining the $p$th root of an element of $K$. The latter theorem builds only the subfield corresponding to a Galois subgroup whose index is a prime number $p$; this subfield is obtained by adjoining a $p$th root to the base field $K$. The proof of this theorem provides details about how we choose this $p$th root (Edwards [30, p.62, 1.1–1.5; l.11–1.26]). In contrast, the proof given in van der Waerden [102, vol.1, p.156, the fundamental theorem] looks empty and impractical. For example, if $\Sigma$ contains an infinite number of elements, it would be impossible to find the corresponding Galois subgroup in a finite number of steps. If we desire to effectively operate a subfield, our focus should be placed on the primitive element rather than all the elements of the subfield.

**Example 3.10.** (Reduction from a large number of cases to a smaller one)

I. $(e_i \wedge e_j) \cdot (e_k \wedge e_l) = \begin{vmatrix} e_i \cdot e_k & e_j \cdot e_k \\ e_i \cdot e_l & e_j \cdot e_l \end{vmatrix}$ [Carmo [18, p.13, 1.14]].

**Proof.** There are $3^4 = 81$ cases to consider.

If $i = j$ or $k = l$, the proof is trivial. Thus, it suffices to consider the cases when $i \neq j$ and $k \neq l$.

If the equality holds for $(i, j, k, l)$, then its holds for $(j, i, k, l)$, $(i, j, l, k)$, and $(j, i, l, k)$ too. Thus, it suffices to consider the cases when $(i, j), (k, l) \in \{(1, 2), (1, 3), (2, 3)\}$.

If $(i, j) = (k, l)$, the proof is trivial; if the equality holds for $(i, j, k, l)$, then its holds for $(k, l, i, j)$. Thus, it suffices to consider the following three cases: $(i, j) = (1, 2), (k, l) = (1, 3); (i, j) = (1, 2), (k, l) = (2, 3); (i, j) = (1, 3), (k, l) = (2, 3).

II. $(e_i \wedge e_j) \wedge e_k = (e_i \cdot e_k) e_j - (e_j \cdot e_k) e_i$ [Carmo [18, p.14, 1.3]].

**Proof.** There are $3^3 = 27$ cases to consider.

If $i = j$, the proof is trivial. Thus, it suffices to consider the cases when $i \neq j$.

If the equality holds for $(i, j, k)$, then its holds for $(j, i, k)$.

22
If \( k \notin \{i,j\} \), the proof is trivial. Thus, it suffices to consider the cases when \( k \in \{i,j\} \). Namely, \((i, j, k) \in \{(1,2,1), (1,2,2), (1,3,1), (1,3,3), (2,3,2), (2,3,3)\}\).

**Example 3.11.** (Reduction: Using the Lorentz condition to separate variables \(A, \phi\) in Maxwell’s equations)

I. Electromagnetism in terms of scalar and vector potentials

By substituting Wangsness [106, (22-1), (22-3)] into Wangsness [106, (21-30), (21-33)], we obtain Wangsness [106, (22-4), (22-5)].

The difficulty with Wangsness [106, (22-4), (22-5)]: Wangsness [106, p.364, l.5–l.10].

When the generalized potentials [Wangsness [106, (22-4), (22-5)]] are applied to the static case, they reduce to the previous ones [Wangsness [106, (11-1), (10-38), (16-18)]].

II. Simplification for l.i.h. materials

Maxwell’s equations for this case: Wangsness [106, (21-42)–(21-45)].

By substituting Wangsness [106, (22-1), (22-3)] into Wangsness [106, (21-42)–(21-45)], we obtain Wangsness [106, (22-8), (22-10)].

The choice of Lorentz condition enables us not only to separate variables \(A, \phi\) [see Wangsness [106, (22-12), (22-13)]], but also to reduce Maxwell’s equations to a single type of differential equation [Wangsness [106, p.365, l.15]].

III. The case for a nonconducting medium

Wangsness [106, (22-11)–(22-13)] reduce to Wangsness [106, (22-14)–(22-16)].

**Example 3.12.** (Reduction of calculations; quality of the output)

The method given in van der Waerden [102, vol.1, p.174, Lemma] is less effective than that given in Edwards [50, p.25, l.–8]. This is because the latter method produces an exact formula. For radicals, all the latter method requires is to take the 10th root of \(t^{10}\), while the former method requires many extractions of roots. That is, the latter method requires less extractions of roots. This is the advantage of using Lagrange’s resolvents.

**Example 3.13.** (Reduction to a lower-level system: reducing a partial differential equation to a system of ordinary differential equations)

The canonical Euler equations represent the characteristic system associated with the Hamilton–Jacobi equation [Fomin–Gelfand [35, p.90, l.–12–l.–10]].

*Proof.* Read Bendersky [7, p.182, l.–3–p.183, l.–1].

Bendersky [7, p.182, (56)] should be corrected as “\(G(x_1, \cdots, x_m, u, p_1, \cdots, p_m) = 0\), where \(p_i = \frac{\partial u}{\partial x_i}\).”

Bendersky [7, p.183, (57)] follows from Tkachev [101, p.2, l.1–l.9].

Remark. See Sneddon [92, chap. 1, §6(f)].

**Example 3.14.** (Reduction with separation of variables in mind; reductions to lower order systems)

Given two differential equations Marion–Thornton [70, p.253, (7.87) & (7.88)]. We want to express \(\lambda\) in terms of \(\theta\) using a single integration even though Marion–Thornton [70, p.253, (7.88)] is a differential equation of the second order. See Marion–Thornton [70, p.254, (7.93)]. The key to reducing the second order differential equation to the first order one is using Marion–Thornton [70, p.253, (7.90)]. Then we can solve the resulting differential equation by the method of separation of variables [Marion–Thornton [70, p.253, (7.91)]].

Example 3.15. (Systematic reduction of calculations by using the formula for the derivative of a determinant)

In order to prove Watson [109] p.76, (7)–(11), Watson suggests that we express successive derivatives of $J_\nu(z)$ and $Y_\nu(z)$ in terms of $J_\nu(z)$, $J'_\nu(z)$, and $Y_\nu(z)$, $Y'_\nu(z)$ by repeated differentiations of Bessel’s equation. The method he suggests is not efficient for calculation. In fact, only the proof of (9) requires the differentiation of Bessel’s equation. In order to derive the rest of formulae effectively and systematically, we should use the formula for the derivative of a determinant. For example, (7) follows from

$$\begin{vmatrix} J_\nu(z) & Y_\nu(z) \\ J'_\nu(z) & Y'_\nu(z) \end{vmatrix} = \frac{d}{dz} \begin{vmatrix} J_\nu(z) & Y_\nu(z) \\ J'_\nu(z) & Y'_\nu(z) \end{vmatrix} = \frac{d}{dz} \begin{vmatrix} J_\nu(z) & Y_\nu(z) \\ J'_\nu(z) & Y'_\nu(z) \end{vmatrix}$$

[Coddington–Levinson [22] p.28, 1.16–1.17].

Example 3.16. (Analogy; reduction of calculations)

Watson–Whittaker [108] §15.8 discusses three theorems in the following order:


b Watson–Whittaker [108]. p.329, l.11–l.9, Theorem (I)

c Watson–Whittaker [108]. p.329, l.7–l.5, Theorem (II)

In my opinion, following the above order is a bad approach. It is better if we discuss c first. This approach will enable us to quickly establish a relationship between $C_{n-1/2}$ and $P_n$. We can prove the equality given on Watson–Whittaker [108] p.329, l.5] using Hobson [53, p.189, (12)] and Guo–Wang [46, p.276, (10)]. Without using c it is difficult to prove a and b if one tries to follow the proof patterns given in Watson–Whittaker [108] p.303, l.15–l.9; p.304, l.6–l.13]. This is because $C_{n-1/2}$ differs from $P_n$ by a factor containing $(z^2 - 1)^{-1/2}$. The difference will cause the number of terms to explore when we use differential operators (Watson–Whittaker [108] p.304, 1.9). It may also cause other problems when we compare coefficients (Watson–Whittaker [108] p.303, l.13]). However, once c is proved, the proof of a and b will become easy. This is because there are corresponding properties of $P_n$ ready for use. a follows from c, Guo–Wang [46, p.250, (16)] and Watson–Whittaker [108, p.326, 1.11]). b follows from c, Guo–Wang [46, p.256, (8)] and Watson–Whittaker [108]. p.324, 1.18]). The formula given in Watson–Whittaker [108 p.324, 1.18] is based on Ferrers’ definition (Watson–Whittaker [108, p.323, l.5]). When we apply the equality to the proof of b we must add a factor of $(-1)^{1/4}$ to the right-hand side of the equality. This is because the notation $P_n$ given in Watson–Whittaker [108 p.329, l.5] is based on Hobson’s definition (Watson–Whittaker [108, p.325, l.3]) instead of Ferrers’ definition (Watson–Whittaker [108 p.323, l.5]).

Example 3.17. (The method of analogical correspondences often proposes conjectures)

By the method of analogical correspondences between the LC system and the mass-spring system given in Halliday–Resnick [47] p.627, 1.9–l.8, the formula $\omega = \sqrt{\frac{k}{m}}$ proposes the conjecture $\omega = \frac{1}{\sqrt{LC}}$ [Halliday–Resnick [47] p.627, (35-3)]. By the method of analogical correspondences between the two differential equations given in Halliday–Resnick [47] p.623, (35-5); (14-6), we obtain Halliday–Resnick [47] p.628, (35-6)] from Halliday–Resnick [47] p.628, (14-8). By substituting Halliday–Resnick [47] p.628, (35-6)] into Halliday–Resnick [47] p.628, (35-5), we prove Halliday–Resnick [47] p.627, (35-3)]. Thus, the method of analogical correspondences often proposes conjectures, while the detailed and complete analogy in terms of mathematics may lead to their proofs. In Example 6.67 I, Table B, entry (3,2) says that the magnetic force derived from energy changes agrees with Ampère’s law. However, I cannot find a corresponding statement
for the parallel-plate capacitor in Wangsness [106]. I fill entry (3, 1) after finding the corresponding statement on the internet.

**Example 3.18.** (Effectiveness: quantitative vs. qualitative formulations)
(The inverse function theorem)
Hartman [48, p.11, Exercise 2.3] provides a quantitative formulation about the inverse function theorem because it assigns the size of the ball \(D_1\) on which \(f\) is one-to-one. In contrast, Rudin [86, p.193, Theorem 9.17] provides only a qualitative formulation because it is stated in terms of open sets. The former is a more effective formulation, because its solution is more informative.

Remark 1. (Proof of Hartman [48, p.11, Exercise 2.3]). By Hartman [48, p.10, Theorem 2.1], there exists a function \(g: \{x: |x| \leq b/M\} \rightarrow D\) such that \(g \circ f = Id\). Similarly, there exists a function \(h: y: |y| \leq b/(MM_1) \rightarrow \tilde{D}_0\) such that \(h \circ g = Id\). \(h = h \circ (g \circ f) = (h \circ g) \circ f = f\) on \(\{y: |y| \leq b/(MM_1)\}\). It is unnecessary to use Hartman [48, p.5, Theorem 2.5].

Remark 2. The inverse function theorem provides a vantage point for us to see the insight of reason why uniqueness implies continuity (Coddington–Levinson [22, p.23, l.10–l.12]).

**Example 3.19.** (Effectiveness: tailoring the solution to the problem’s needs)
The first solution [Marion–Thornton [70, p.422, l.9–p.423, l.3]] is based on Marion–Thornton [70, p.77, (2.83); p.421, (11.21)]. The second solution [p.423, l.2–p.424, l.6] is based on the Lagrangian equation of motion [Marion–Thornton [70, p.231, (7.4)]. Since Marion–Thornton [70, p.77, (2.83)] is more specific than Marion–Thornton [70, p.231, (7.4)], the first solution is more effective. In contrast, Marion–Thornton [70, p.231, (7.4)] is more general and focuses on only one thing – the general method of finding the equation of motion, so when we use the Lagrangian method, the meaning of everything else about the method becomes more ambiguous to the original problem. Consequently, Marion–Thornton [70, p.231, (7.4)] should not be used for understanding the surroundings of the solution.

**Example 3.20.** (Effectiveness: avoiding unnecessary complications)
The use of a complicated theorem can make a constructive existence less effective. Sometimes, it is impossible for a generalized theorem to preserve the effectiveness of a specific case.
(The Riemann–Lebesgue Lemmas)
Suppose we want to use a computer to verify Watson–Whittaker [108, p.172, Theorem 9.41 (I)] for a given function. It is easier to effectively convert the argument to a computer program using the proof of Watson–Whittaker [108, p.172, Theorem 9.41 (I)] than it is to do so using the proof given in Rudin [88, p.109, l.16–l.6]. This is because a complicated theorem (Rudin [88, p.96, Theorem 4.25]) is used in Rudin [88, p.109, l.13]. In addition, the method given in Rudin [88, §5.14] cannot be used to prove the specific case given in Watson–Whittaker [108, p.172, Theorem 9.41 (II)].

**Example 3.21.** (Effectiveness: how generalization affects effectiveness)
The concept of Galois resolvent can be generalized to that of a primitive element (Edwards [30, p.46, Exercise 13 & 14]). For each statement about the Galois resolvent in Edwards [30, §38–§41], we can replace the field generated by the Galois resolvent with the field generated by a primitive element. If we use the definition of the Galois group given in Edwards [30, p.51, l.10–l.23], we must prove the following statements: Edwards [30, p.53, l.11–l.8; p.51, l.7–l.6]. If we use the definition of the Galois group given in van der Waerden [102, vol.1, p.154, l.10 & l.24], we need not prove the above statements.

**Example 3.22.** (Effectiveness: take full advantage of available resources)
Through application we may gain a vantage-point for effectiveness because there are more resources
available. Effectiveness is surely an ongoing trend for the development of mathematics.

We use reduction to absurdity to prove “path-connectedness ⇒ connectedness” (Dugundji [28, p.115, Theorem 5.3]), so it is more straightforward and effective to check connectedness by finding a path between two points. Even though the formal structure for connectedness is firmly established (Chevally [19, p.36, Proposition 2]), we prefer proving connectedness by operating explicit paths (Fomenko [34, p.14, l.6; p.15, l.—2]) rather than by deducting from ineffective structure theorems (Chevally [19 p.37, l.2–l.4]).

Example 3.23. (Effectiveness; insightfulness: perspectives)

When we talk about effectiveness, we must know what aspect we refer to. Suppose we want to express a function as an infinite product. Sometimes, we refer to an entire process of constructing the expansion. If we want to construct the expansion from scratch, then Guo–Wang [46, p.25, Theorem 1] is more effective than González [43] p.202, Theorem 3.16]. However, if the identity for the expansion is already given (González [43] p.206, (3.5-7)), we just want to prove its validity, then the less effective and more general theorem (González [43] p.202, Theorem 3.16]) can be the best choice. Compare the proof of González [43] p.206, (3.5-7]) with that of Guo–Wang [46, p.26, (2)].

Example 3.24. (Hitting multiple targets with one shot)


Example 3.25. (Hitting multiple targets with one shot)

Birkhoff uses Birkhoff–Rota [10, p.25, (23)] to simultaneously prove both the uniqueness and the continuous dependence on initial values.

Example 3.26. (Hitting multiple targets with one shot)

Hartman uses Hartman [48, p.9, l.—1] to prove the uniqueness and estimate the error term at the same time.

Example 3.27. (Hitting multiple targets with one shot)

Rudin [88, p.130, l.18].

Example 3.28. (Reduction of calculations; directness; simplicity)

It is simpler to use Ince [54, p.161, l.21–l.23] rather than Watson–Whittaker [108, p.202, l.16] to find out whether $z = \infty$ is a regular singular point of the second-order ODE.

Example 3.29. (Simplicity)

(The root test vs. the ratio test)

For convergence tests, we choose the ratio test for $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, and choose the root test for $\sum_{n=0}^{\infty} \left(\frac{2n^2 + 1}{n^2 + 1}\right)^n$.

Example 3.30. (Simplicity; directness; avoiding unnecessary complications)

(The Bessel function of order $n$)

The introduction of the Bessel function of order $n$ given in Ince [54, p.189, l.—2–p.190, l.8] is simple and direct, while the introduction of the same function given in Watson–Whittaker [108, p.355, l.12–p.356, l.5] is unnecessarily complicated.

Example 3.31. (Simplicity)

(Recurrence relations for Bessel functions)
In order to prove the recurrence relations for Bessel functions, it is simpler to use the generating function (Watson–Whittaker [108, p.358, Example 1]) than use the integral representation (Watson–Whittaker [108, §17.21]).

The proofs given in Watson–Whittaker [108, §17.21] use integral representations, while those given in Guo–Wang [46, p.349, l.8–p.350, l.9] use series. The latter method is more elementary, so it is better.

Example 3.32. (Simplicity; directness)

Edwards [30, p.86, l.−7–p.87, l.10] discusses how Gauss and Kronecker proved the following statement:

$$g(a)g(a^2)g(a^{p−1}) ≡ g(1)^p−1 \pmod{p},$$

where $a \neq 1$ and $a^p = 1$. Their ideas might be new during their times. Now we may adopt a simpler, more direct and inspiring method to prove the above statement using the following theorem:

*If an ordinary integer is divisible by $a − 1$, then it is divisible by $p$ (Edwards [29, p.93, l.8–l.10]).*

Example 3.33. (Simplicity; avoiding unnecessary complications)

Given that $P_n(x)$ is a solution of Legendre’s equation, find the second solution.


Example 3.34. (Simplicity: using simple statements to prove complicated ones; avoiding unnecessary complications)

Both van der Waerden [102, vol.1, p.112, I.3–I.19] and Edwards [30, p.142, I.2–I.6] prove the existence of a primitive $(q−1)$st root of unity. The former approach is more direct and constructive because it uses simple statements to prove more complicated ones.

Example 3.35. (Simplicity; avoiding unnecessary complications)

In order to include all the solutions (Coddington–Levinson [22, p.115, I.−11–I.−10]), Levinson considers formal Laurent series (Coddington–Levinson [22, p.116, I.12]). In fact, through the transformation given in Hartman [48, p.79, (11.20)], it requires only the consideration of the formal power series (Hartman [48, p.80, I.14; p.78, Theorem 11.3]) rather than that of formal Laurent series (Coddington–Levinson [22, p.117, Theorem 3.1]). The complications given in Coddington–Levinson [22, p.115, I.−11–p.118, I.24] are unnecessary.

Example 3.36. (Advantages: taking full advantage of circumstances; reduction of calculations)

We should take advantage of the situation whenever possible. Given a polynomial $P$ of degree $n$. We would like to express a symmetric polynomial of roots of $P$ in terms of the coefficients of $P$ (Jacobson [56, vol. 1, p.109, Theorem 9]). For example, we want to find $n^2$ in terms of $\sigma_1, \sigma_2, \sigma_3$ (Edwards [30, p.22, Exercise 2]). First, we need a reference table such as the one given in Edwards [30, pp.6–7, (1)–(19)]. The method given in Edwards [30, pp.14–15, Exercise 12] applies to the general case. However, it would be ineffective to use this term by term transformation to solve our specific case. We should try to group terms and express them in terms of $\sigma_1, \sigma_2, \sigma_3$ whenever possible before we carry out our calculations. For example, the following observation saves us tremendous calculations:

$$x^2y + y^2z + z^2x + y^2x + z^2y + x^2z)^2 - 2(x^3y + y^3z + z^3x) - 6x^2y^2z^2 - 2(xyz)(x^3 + y^3 + z^3).$$
Example 3.37. (Advantages: shortcuts; effectiveness; reduction of calculations)

\[ \frac{d^2 \theta}{d\theta^2} \sum_{m=0}^{\infty} \frac{(-1)^m e^{2m \sin \theta}}{(2m)!} \bigg|_{\theta=0} = \frac{d^2 \theta}{d\theta^2} \sum_{m=0}^{p} \frac{(-1)^m e^{2m \sin \theta}}{(2m)!} \bigg|_{\theta=0} \ \text{[Watson [109, p.36, 1.7–1.10]].} \]

Proof. Because we only need to consider the \( \theta^{2p} \) term, we may reduce the index set of summation from the infinite set \( \{0, 1, 2, \cdots \} \) to the finite set \( \{0, 1, 2, \cdots, p \} \). \( \square \)

Example 3.38. (Flexibility)

Levinson proves Coddington–Levinson [22, p.20, Theorem 5.1] using the exact form of the general theorem (Coddington–Levinson [22, p.12, Theorem 3.1]). However, the domain of the solution in the general case is not large enough to meet the requirement. Consequently, he uses Coddington–Levinson [22, p.15, Theorem 4.1] to extend the domain. In contrast, the proof of Pontryagin [81, p.167, l.–10–p.169, l.–1] applies the same idea directly to the entire domain. Consequently, it is immune from the fuss of domain expansion. The techniques of domain expansion given in Coddington–Levinson [22, p.15, Theorem 4.1] and Coddington–Levinson [22, p.20, Theorem 5.2] are unnecessary and insignificant.

Example 3.39. (Avoiding contradictions: from the Galilean invariance to the Lorentz invariance)

The Michelson–Morley experiment [Fowler [36]] suggests that the velocity of light be constant, independent of any relative motion of the source and the observer. However, the Galilean transformation is inconsistent with this suggestion [Marion–Thornton [70, p.548, l.–14–l.7]]. Maxwell’s equations are invariant in form under the Lorentz transformation [Wangsness [106, l.29–5]]. For the contradiction caused by the Galilean invariance given in Marion–Thornton [70, p.549, l.–9–l.11], the solution is to use Lorentz transformations instead of Galilean transformations. Muon decay provides an experimental verification of special relativity [Marion–Thornton [70, p.555, l.–14–p.556, l.–16]]. The speed of the light in invariant under Lorentz transformations [Marion–Thornton [70, p.552, l.1–l.5]]. Linear momentum is not conserved according to special relativity if we use the conventions for momentum of classical physics [Marion–Thornton [70, p.564, l.1–l.6]]. The key to solving this inconsistency is to use the proper time in the definition of linear momentum [Marion–Thornton [70, p.564, l.21–l.26; p.565, Example 14.6]]. Marion–Thornton [70, p.567, Example 14.7] shows that the relativistic kinetic energy reduces to the classical result for small speeds, \( u \ll c \).

If we use the position 4-vector \( \mathbf{x} \) with \( x_4 = i ct \) to construct the Lorentz transformation matrix [Marion–Thornton [70, p.572, (14.77)]], then the momentum vector \( p \) given in Marion–Thornton [70, p.564, (14.45)] becomes the momentum–energy 4-vector \( P = (p, i E) \) [Marion–Thornton [70, p.573, (14.91)]], where \( E \) is the total energy. The contradiction given in Marion–Thornton [70, p.574, l.–7–l.–1] forces us to modify the velocity addition rule: Marion–Thornton [70, p.576, (14.98)]. In order to make the Lagrange equations [Marion–Thornton [70, p.578, (14.107)]] accommodate the relativistic momentum vector [Marion–Thornton [70, p.578, (14.108)]]], we must modify the definition of Lagrangian [Marion–Thornton [70, p.578, (14.113)]] because mass and energy are interrelated in relativity theory; it no longer is meaningful to speak of a “center-of-mass” system; in relativistic kinematics, one uses a “center-of-momentum” coordinate system instead. Such a system possesses the same essential property as the previously used center-of-mass system—the total linear momentum in the system is zero [Marion–Thornton [70, p.579, l.–16–l.–11]]. This modification of coordinate system leads to Marion–Thornton [70, p.582, (14.128); (14.129)] which are reduced to the classical results given in Marion–Thornton [70, p.350, (9.69); (9.73)] when \( \gamma \to 1 \).

Example 3.40. (Expanding the scope of application without loss of efficiency: methods of determining the stability of circular orbits) [Marion–Thornton [70, §8.10]]

Marion–Thornton [70, p.317, l.5–l.7] gives a method of determining the stability of circular orbits. However, examples that may take advantage of this method are few. They are limited to simple functions
such as that given in Marion–Thornton [70] p.317, (8.75)). For the complicated function given in Marion–Thornton [70] p.319, (8.96)], the method would take a tremendous amount of calculations. Instead, we should use the method given in Marion–Thornton [70] p.317,1.19–p.319,1.−8].

Example 3.41. (Structurization: the critical structure may fail to emerge more often because of lacking in skilful analysis)


Proof 1.

\[
\begin{align*}
\sum_{m \leq i \leq n} r \delta_{i1} \bar{r} \delta_{i1} & \quad \sum_{m \leq i \leq n} r \delta_{i1} \bar{r} \delta_{i2} \\
\sum_{m \leq i \leq n} r \delta_{i2} \bar{r} \delta_{i1} & \quad \sum_{m \leq i \leq n} r \delta_{i2} \bar{r} \delta_{i2}
\end{align*}
\]

\[
= \sum_{m \leq i \leq n} \sum_{m \leq j \leq n} r \delta_{i1} r \delta_{j2} \bar{r} \delta_{i1} \bar{r} \delta_{j1} - \sum_{m \leq i \leq n} \sum_{m \leq j \leq n} r \delta_{i1} r \delta_{j2} \bar{r} \delta_{i2} \bar{r} \delta_{j1}
\]

\[
= \sum_{m \leq i < j \leq n} r \delta_{i1} r \delta_{j2} (\bar{r} \delta_{i1} \bar{r} \delta_{j2} - \bar{r} \delta_{i2} \bar{r} \delta_{j1}) + \sum_{m \leq j < i \leq n} r \delta_{i1} r \delta_{j2} (\bar{r} \delta_{i1} \bar{r} \delta_{j2} - \bar{r} \delta_{i2} \bar{r} \delta_{j1}).
\]

\[
\begin{bmatrix}
    r_1 \bar{r}_1 & r_1 \bar{r}_2 \\
r_2 \bar{r}_1 & r_2 \bar{r}_2
\end{bmatrix}
= \begin{bmatrix}
r_1 & 0 \\
r_2 & 0
\end{bmatrix}
\begin{bmatrix}
\bar{r}_1 & 0 \\
0 & 0
\end{bmatrix}.
\]

Proof 2. Since the sum of positive semidefinite matrices is positive semidefinite, it suffices to observe

\[
\begin{bmatrix}
r_1 \bar{r}_1 & r_1 \bar{r}_2 \\
r_2 \bar{r}_1 & r_2 \bar{r}_2
\end{bmatrix}
= \begin{bmatrix}
r_1 & 0 \\
r_2 & 0
\end{bmatrix}
\begin{bmatrix}
\bar{r}_1 & 0 \\
0 & 0
\end{bmatrix}.
\]

Remark. The second proof can be generalized to prove Coddington–Levinson [22] p.263, 1.4, (ii).

Example 3.42. (Insightfulness)

Suppose we want to prove

\[
\int_0^\pi \frac{dx}{a+b \cos x} = \int_0^\pi \frac{dx}{\sqrt{a^2 - b^2}} (|b| < |a|).
\]

Proof. Let \( t = \tan \frac{x}{2} \).

\[
\begin{align*}
\frac{dt}{dx} & = \frac{1}{1+t^2} \\
\cos x & = \frac{1-t^2}{1+t^2} \\
\int \frac{dx}{a+b \cos x} & = 2 \int \frac{dt}{(a+b) + (a-b)t^2} \\
& = \frac{2}{\sqrt{a^2 - b^2}} \arctan(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}) + C.
\end{align*}
\]

The above proof by calculus is symbolic and incomplete because it is difficult to determine the value of the integral in some cases. Both Hobson [53] p.360, 1.12–p.361, 1.21] and Guo–Wang [46] p.268, 1.2–1.16] prove the formula for \( \int_0^{2\pi} \frac{dx}{A + B \cos x + C \sin x} \) in all cases. The former proof uses the integral of sin and that of cos, while the latter proof uses the residue theorem. In order to see whether the statement given in Hobson [53] p.361, 1.18–1.19] is true, we have to do some calculations from the viewpoint of the former proof. However, from the viewpoint of the latter proof, we can see the reason directly (Guo–Wang [46] p.268, 1.12–1.16]). Consequently, the latter proof is well-structured and insightful.
Example 3.43. (Insightfulness; accessibility: narrowing the scope of mathematical induction; essence)

When we try to solve a problem, we should focus on the essence of the problem. Saks–Zygmund [90, p.108, Theorem 3.4] is a corollary of the general theorem given in Saks–Zygmund [90, p.107, Theorem 3.1]. The latter theorem specifies the conditions under which we may differentiate under the integral sign. Thus, the latter theorem highlights the essence of Saks–Zygmund [90, p.108, Theorem 3.4]. In contrast, Ahlfors [1, p.121, Lemma 3] provides only a trick to solve a particular problem. After studying Ahlfors’ proof, we do not know the general method of solving this type of problem.

Another drawback of Ahlfors’ proof is that the scope of mathematical induction used in his proof is too broad. We should limit the scope of mathematical induction used in a proof as narrowly as possible. In the proof of Saks–Zygmund [90, p.108, Theorem 3.4], the mathematical induction is used to derive

\[ \frac{d}{dz} \frac{z^{(k+1)}}{z^k} = -(k+1)z^{-(k+2)}. \]

In contrast, Ahlfors tries to justify his differentiation in each induction step (Ahlfors [1, p.121, 1–8–p.122, 1.8]). In other words, he differentiates under the integral sign countable times. On the one hand, it takes too much time figuring out trivial details; on the other hand, it consumes too much time and memory for computer if we allow too much work for each induction step.

Example 3.44. (Insightfulness: formal solutions)

In Guo–Wang [46, p.81, l.12], we assume that the interchange of the integral sign and \( L \) is valid. This procedure allows us to quickly obtain a formal solution \( u(z) \) (Guo–Wang [46, p.81, (5) & (6); p.82, (7)]). The assumption will be justified later case by case. For example, in order to prove both that Guo–Wang [46, p.302, (2)] satisfies Guo–Wang [46, p.302, (1)] and that the integral given in Watson–Whittaker [108, p.339, 1–5] satisfies Watson–Whittaker [108, p.337, (B)], we use Rudin [88, p.27, Theorem 1.34] to justify the differentiation under the integral sign.

After we obtain a formal solution, it is easy to forget to prove it to be a true solution. In Guo–Wang [46, §6.4], Guo fails to rigorously prove that the formal solution given in Guo–Wang [46, p.302, (2)] is a solution of Guo–Wang [46, p.302, (1)]. Since the integral given in Guo–Wang [46, p.305, (1)] and the left-hand side of the equality given in Guo–Wang [46, p.305, (2)] are obtained by replacing \( \alpha \) in Guo–Wang [46, p.302, (2) and (3)] by \(-t\), Guo–Wang [46, p.305, (1)] can only be considered a formal solution of the Whittaker equation (Guo–Wang [46, p.300, (1)]). Guo also fails to rigorously prove that this formal solution is indeed a solution. In contrast, Watson–Whittaker [108, p.339, 1–6–p.340, L6] rigorously prove that the integral given in Watson–Whittaker [108, p.339, 1–5] is a solution of the Whittaker equation (Watson–Whittaker [108, p.337, (B)]). Note that Watson leaves out a factor, \((-1)^{-k-1/2+m}\), on the right-hand-side of the equality given in Watson–Whittaker [108, p.340, 1.2–1.3].

Example 3.45. (Insightfulness: formal solutions)

We can quickly derive Guo–Wang [46, p.298, (6)] by replacing \( z \) in Guo–Wang [46, p.143, (10)] by \( z/\beta \), and letting \( \beta \to \infty \). This formal procedure is justified in Guo–Wang [46, p.302, 1.2–p.303, 1.5].

Example 3.46. (Insightfulness: formal solutions)

We can quickly derive Guo–Wang [46, p.303, (6)] from Guo–Wang [46, p.153, (7)] by interchanging \( \alpha \) and \( \beta \), replacing \( z \) by \( z/\beta \), and letting \( \beta \to \infty \). This formal procedure is justified in Guo–Wang [46, p.303, 1.10–1.17].

Example 3.47. (Insightfulness: perspectives)

Let \( m \leq [n/2] \). Then

\[ \sum_{k=m}^{[n/2]} \binom{n}{2k} \binom{k}{m} = \frac{2^{n-2m-1}n!(n-m-1)(n-m-2)\cdots(n-2m+1)}{m!}. \]

**Proof.** Using the expansion of \((1+x)^n\) and then letting \( x = 1 \) or \(-1\), we can prove case \( m = 0 \). Using the expansion of \( \frac{d}{dx}(1+x)^n \) and then letting \( x = 1 \) or \(-1\), we can prove case \( m = 1 \). However, it will become
difficult to prove case \( m \geq 2 \) if we continue to use the above combinatorial method. We shall resort to Bessel functions.

\[
(\sinh \theta + \cosh \theta)^n + (\sinh \theta - \cosh \theta)^n = \sum_{k=0}^{[n/2]} \sum_{r=0}^{k} \binom{n}{2k} \binom{k}{r} \sinh^{n-2k+2r} \theta.
\]

The result follows from Watson–Whittaker [108, p.375, l.18–l.20] and Watson [109, p.272, (4)]. The properties of Bessel functions match our needs naturally and perfectly just as the properties of the Riemann zeta function match the needs for proving the prime number theorem.

**Example 3.48.** (Insightfulness)

Edwards [30, p.59, l.22–l.28] and van der Waerden [102, vol.1, p.158, l.1–l.8] both describe how the extension \( K' \) of the base field \( K \) affects the Galois group of a polynomial equation \( f = 0 \). In contrast, the former approach is more insightful.

Example 3.49. (Insightfulness: only through studying the advanced theory may we master the basic one)

One cannot master calculus unless one completes one’s study in advanced calculus. One cannot master Coddington–Levinson [22, chap. 1] until he fully understands Coddington–Levinson [22, chap. 7, p.189, Theorem 2.1]. A topic represents merely a stage of a theory’s development. If one understands every word in a textbook about a topic, it does not mean one masters the topic. This is because one understands the theory up to that topic, but has not applied the topic to the later part of the theory.

Example 3.50. (Essence; directness; simplicity)

(How Lagrange solved \( Pp + Qq = R \) [Boole [12, p.318, l.2–p.319, l.11]])

Lagrange’s original solution. \( dz = pdx + qdy \).

\( Pdz - Rdx = q(Pdy - Qdx) \).

Suppose \( Pdz - Rdx = du \) and \( Pdy - Qdx = dv \). Then \( du = qdv \).

Since the left side of the equality is an exact differential, \( q = \varphi'(v) \), where \( \varphi'(v) \) is an arbitrary function of \( v \).

By separation of variables, we have

\[
u = constant = \phi(v).
\]

\[
du = 0 = \varphi'(v)dv = 0.
\]

\[ (du = 0 = dv) \leftrightarrow (\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}). \]

Let the solutions of \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \) be solved for the constants of integration, thus

\[ u_1(x,y,z) = a, \quad v_1(x,y,z) = b \] [Ince [54, p.47, l.−14]]. Then we identify \( u, v \) with the constants of integration \( a, b \).

Remark. Shall we assign \( u_1 \) to \( u \) or \( v \)? According to the above considerations, \( u \) and \( v \) are not symmetric. However, according to the analytic proof given in Sneddon [92, p.52, l.9–l.10], \( u \) and \( v \) are symmetric [Sneddon [92, p.52, l.12]]. Consequently, it doesn’t matter which one \( u_1 \) should be assigned to. Sneddon [92, p.50, l.10–p.52, l.7] gives a geometric proof about the equivalence between \( Pp + Qq = R \) and \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \). However, among the three (original, analytic, and geometric) proofs, Lagrange’s original proof shows most clearly the key to solving \( Pp + Qq = R \). Lagrange goes straight to the heart of the matter. In contrast, Sneddon seems to focus on its side problems; his approach may easily make readers unable to see the wood for the trees.
Example 3.51. (Essence: seeking the common pattern of solutions; accessibility)
(The Lagrange resolvent)

If we can solve problems for some specific cases, we would like to find a common method for the general case (Edwards [30], p.2, l.23–l.28). According to Edwards [30], p.17, (1); p.20, l.4–l.7; p.21, l.2–l.5], Lagrange observed that all the roots can be expressed in terms of a resolvent and its conjugates. After analyzing his observations, he proved the theorem given in Edwards [30, p.33, l.1–11; p.34, l.10–l.17] using Lagrange resolvents to solve a polynomial equation with low degree and multiple roots. The following path shows the evolution of Lagrange’s resolvents: Edwards [30, (p.22, l.16 & l.13) → (p.25, (1)) → (p.29, l.19)]; the right-hand side of the arrow is more organized than the left-hand side. Galois went further to associate the roots of a polynomial equation with the group of transformations that permute the roots. By creating the Galois resolvent (Edwards [30, p.114, l.10–l.22]), he was able to use it to generate the splitting field \( K(a, b, c, \cdots) = K(t) \) of a polynomial equation \( f(x) = 0 \), where \( a, b, c, \cdots \) are roots of \( f(x) = 0 \) (Edwards [30, p.114, l.11–l.22]). That is, he expressed the splitting field as a simple algebraic extension \( K(t) \) of \( K \).

Example 3.52. (Essence; insightfulness; accessibility; effectiveness: avoiding trial and error; networks: interactions)
(The Galois resolvent)

For the concepts of a group (Edwards [30, p.48, l.12–l.8; p.49, l.17–l.19]), a subgroup (Edwards [30, p.50, l.1–l.2]), or a normal subgroup (Edwards [30, p.50, l.17–l.20]), Galois approached them from two viewpoints: the viewpoint of the roots of the polynomial and the viewpoint of the group itself. The former viewpoint is natural, concrete, as well as insightful and provides an easy way to find the elements of the group. It allows interface. In contrast, the latter viewpoint is abstract, but provides effective methods to check whether these elements satisfy the conditions in a definition or prove the properties of a definition in an organized manner (Edwards [30, p.49, l.22–l.15; p.56, Exercise 1, 2, and 3]). In other words, the former viewpoint involves concrete transformations that permute roots. Consequently, we have concrete resources to work with. Nowadays, the definition of a group and that of a normal subgroup in most textbooks lack origins and resources, especially significant examples for students to gain hands-on experience for these concepts. When we try to solve a polynomial equation, we should directly work with the permutation group of roots rather than the general concepts of group theory. For example, although both Edwards [30, p.51, l.22–l.13] and van der Waerden [102, vol.1, p.154, l.7–l.10] discuss the Galois group, the former discussion is more intuitive and to the point. This is because in Edwards [30, p.51, l.22–l.13] the Galois resolvent is used to define the Galois group directly in terms of conjugates (Edwards [30, p.51, l.18]), while in Jacobson [56] vol. 3, p.27, l.8–l.7 or van der Waerden [102] vol.1, p.154, l.7–l.13 & l.24] the Galois group is defined in terms of invariants (Jacobson [56] vol.3, p.27, l.15 & l.28–l.29; van der Waerden [102 vol.1, p.154, l.9–l.10]). If the subfield has an infinite number of elements, then it is difficult to check the latter definition in a finite number of steps. Thus, the former definition is more accessible than the latter definition. Galois proved the latter definition as a theorem (Edwards [30, p.52, l.10–l.9]). The latter definition is useful in the generalization from the Galois group of a polynomial equation to the Galois group of a normal field with respect to the base field (van der Waerden [102 vol.1, p.154, l.10–l.11]).

Similarly, it is more natural, consistent (Edwards [30, p.84, l.8–l.4]), advantageous (Edwards [30, p.120, l.23–l.14]) to define a subgroup using the Galois resolvent than using group theory. It would make its meaning rich and its concept clear if we define a normal subgroup (Edwards [30, p.51, l.3–l.6]) using the Galois resolvent rather than group theory.

Remark 1. Although Lagrange’s methods of selecting the roots of resolvent equation (Edwards [30], p.18,
Example 3.54. (Essence: finding the cases that Riccati’s equation is integrable in finite terms; insightfulness)

If Riccati’s equation \( \frac{dy}{dx} = az^n + by^2 \) is integrable in finite terms, then \( n = -2 \) or \( -\frac{4m}{2m+1} \) (\( m = 0, 1, 2, \cdots \)) [Watson [109] p.123, l.23–1.26]. Since Riccati’s equation is a variant [Ince [54] p.24, l.11–p.25, l.14]: Watson [109] p.96, (6)] of Bessel’s equation, we may use the language of Bessel’s functions to translate the above theorem as follows:

If Bessel’s equation for functions of order \( \nu \) [Watson [109] p.117, l.10] is soluble in finite terms, then \( 2\nu \) is an odd integer.

Remark 2. van der Waerden [102] vol.1, p.181, l.4–l.10; p.191, l.4–l.2] claims, “Each single \( \Theta \) is fixed by 8 permutations; the three of them together remain fixed only under \( B_4 \).” One may argue, “An automorphism \( \sigma \) fixing \( \Theta_1 \) will fix \( \Theta_2 \), because \( \Theta_2 \in \Delta(\sqrt{D})[\Theta_1] = \Delta(\sqrt{D})(\Theta_1, \Theta_2, \Theta_3) \). This would contradict van der Waerden’s claim.”

Clarification to the above confusion. \( \sigma \) has the above property if \( \sigma \) belongs to the Galois group of \( \Delta(\sqrt{D})[\Theta_1]/\Delta(\sqrt{D}) \). However, the 8 permutations belong to the Galois group of \( \Sigma/\Delta \). If \( \tau \) belongs to the Galois group of \( \Sigma/\Delta \), \( \tau \) is determined only if its value at the generator of \( \Sigma/\Delta \) is determined. \( \Theta_1 \) is the generator of \( \Delta(\sqrt{D})[\Theta_1]/\Delta(\sqrt{D}) \), but is not the generator of \( \Sigma/\Delta \).


Example 3.53. (Essence: proving insolvability by locating the first obstacle to solution; insightfulness)

(The general equation of degree \( n > 4 \) is not solvable by radicals)

In order to prove a problem is insolvable with assigned tools, we first proceed to find a solution until we meet obstacles. The lessons of failure may inspire us to approach the problem differently. In order to prove Edwards [30] p.91, l.7–l.8, Corollary], we must investigate the solutions of a cubic equation. The feature of the method given in Edwards [30] l.14] is the use of Lagrange’s resolvents. All the roots can be expressed in terms of Lagrange’s resolvents. The organized solution given in Edwards [30] p.133, l.9–l.134, l.1–l.1] shows that if the Galois group \( G \) is solvable (Edwards [30] p.61, l.16]), we may divide the process of finding solutions into \( n \) steps. Each step corresponds to a specific subgroup and thereby a specific subfield. In each step, Edwards [30] p.61, Proposition] provides a tool that we may use to proceed toward our goal. This structured solution indicates that if the Galois group is not solvable, we can locate the step in which we will encounter the first obstacle to the solution. Since the circumstances in the beginning of meeting obstacles is less complicated than those in later stages, it is easier for us to analyze and figure out the reason why the equation is not solvable. Intermediate field extensions were used by Gauss to simplify the construction of a \( p \)th root of unity (Edwards [30] p.29, l.1–l.15; p.30, l.12]). Galois’ contribution was to find a subgroup corresponding to each subfield (Edwards [30] p.57, l.18–l.20]).

Remark. The same idea can be used to prove that it is impossible to solve the equation \( x^3 + px + q = 0 \) by real radicals for the case \( D > 0 \). See van der Waerden [102] vol.1, p.180, l.11–l.14].
In the proof [Watson [109] §4.7–§4.74] of the latter version, despite possible difficulties, all the problems can be solved except one. That is, an infinite power series cannot be expressed as a polynomial [Watson [109] p.123, l.4]. From this, we may determine the cases that Bessel’s equation is soluble in finite terms [Watson [109] p.123, l.6–l.8].

**Example 3.55.** (Essence: existence and uniqueness of solutions of ODEs)

There are 30 theorems and one corollary in Coddington–Levinson [22, chap.1 & chap.2]. They all discuss the existence and uniqueness of solutions of differential equations; many of them use the same method. Only through organization may we see essential solution strategies clearly.

(1) By treating parameters as initial values or treating initial values as parameters, we may reduce the number of problems from two to one.

By treating parameters as initial values (Coddington–Levinson [22, p.31, l.9–l.10]), Coddington–Levinson [22, Theorem 7.5] can be considered a simple application of Coddington–Levinson [22, Theorem 7.2]. Similarly, Coddington–Levinson [22, Theorem 7.4] can be considered an application of Coddington–Levinson [22, Theorem 7.1]. By treating initial values as parameters (Pontryagin [81, p.178, l.10–l.3], Pontryagin [81, Theorem 15] can be considered a simple application of Pontryagin [81, Theorem 13; p.173, Theorem 14; p.177, (B)].

(2) Generalization for solutions’ continuity with respect to parameters:

(3) Generalization for successive approximations:
The Lipschitz condition

Coddington–Levinson [22, chap.1, sec.3] provides the Lipschitz condition for the generalized Lipschitz condition for Lebesgue integrals:

- Functional values in one section are determined by functional values of the previous section.
- ODE is valid almost everywhere; solution is absolutely continuous.
- Coddington–Levinson [22, p.54, Theorem 3.1] asserts the convergence of successive approximations.

Continuity

Coddington–Levinson [22, p.4, Fig.1] uses a polygonal line to approximate the flow of the differential equation.

Example 3.56. (Essence: the frequently used statements in a theory should be considered valued basics)

Coddington–Levinson [22, p.244, (4.10); p.248, l.−6] use the construction given in Birkhoff–Rota [10, p.286, (67)], but the former books never express it as a theorem explicitly.

Example 3.57. (Essence: the essence often becomes clearer if we reduce a complicated case to a simple case)

The proof of Sturm’s oscillation theorem given in Coddington–Levinson [22, p.212, Theorem 2.1] is too complicated to read or understand its essence. The following steps may help recognize its essence:

1. If we consider the solution of an equation of motion as the path of a particle, then the phase plane may offer the detailed behavior of the motion [Pontryagin [81, p.125, Figure 39–Figure 41] and Arnold [4, p.148, Fig. 106]]. If we use the Prüfer substitution, a concept similar to that of the Poincaré phase plane, the differential equation of the second can be reduced to that of the first order [Birkhoff–Rota [10, p.267, (22)] and Birkhoff–Rota [10, p.27, Corollary 1] can be interpreted as Birkhoff–Rota [10, p.269, Lemma 1].

If we are versed in the above basics, we will have no problem studying the proof of Coddington–Levinson [22] p.212, Theorem 2.1. Now let us prove a few statements in this proof:

Statement 1. \( \omega(t, \lambda) > k\pi \) for \( t > t_k \) [Coddington–Levinson [22] p.212, 1.17].

Proof. Since \( \omega'(t_k) > 0 \), there exists an \( \varepsilon > 0 \) such that \( \omega(t, \lambda) > k\pi \) on \( (t_k, t_{k+1}) \). If \( \omega(t, \lambda)(t \in (t_k, t_{k+1})) \) intersects with \( \theta = k\pi \), this will contradict the definition of \( t_{k+1} \).

Statement 2. \( \omega(c, -\lambda) \leq \delta \) for \(-\lambda \) large enough [Coddington–Levinson [22] p.213, 1.15].

Proof. (a) If \( \omega(c, -\lambda) > \pi - \delta \), there exists a \( t_1 \) such that \( \omega(t_1, -\lambda) = \pi - \delta \). By Coddington–Levinson [22] p.213, 1.12, \( \omega(t) \) is decreasing at \( t = t_1 \). Once \( \omega(t, -\lambda)(t_1 < t \leq c) \) intersects with \( \theta = \pi - \delta \), it will start to decrease. Hence, \( \omega(t, -\lambda) \leq \pi - \delta \) on \( (t_1, c] \).

(b) \( \omega(c, -\lambda) \geq 0 \) [Coddington–Levinson [22] p.212, 1.6] and \( \omega(a, -\lambda) = \alpha < \pi - \delta \) \( \Rightarrow \omega(c, -\lambda) - \omega(a, -\lambda) > -\pi \).

(c) Assume \( \omega(c, -\lambda) > \delta \). Then \( \omega(c, -\lambda) - \omega(a, -\lambda) = \omega(t_2, -\lambda)(c-a) \), where \( t_2 \in (a, c) \). By [22] \( \omega(t_2, -\lambda) \leq \pi - \delta \). \( \omega(t_2, -\lambda) \) cannot be less than \( \delta \). Otherwise, \( \omega(t, -\lambda)(t \in (t_2, c]) \) will intersect with \( \theta = \delta \) at \( t = t_3 \). By Coddington–Levinson [22] p.213, 1.13, \( \omega(t) \) will start to decrease at \( t_3 \) and as \( t \) increases whenever \( \omega(t) \) raises to \( \delta \), it will start to decrease. Hence, \( \omega(t, -\lambda) \leq \delta \) on \( (t_3, c] \). But this contradicts our assumption \( \omega(c, -\lambda) > \delta \).

(d) By Coddington–Levinson [22] p.213, 1.14, \( \omega(c, -\lambda) - \omega(a, -\lambda) = \omega(t_2, -\lambda)(c-a) < -10 \). This contradicts [22]

Statement 3. \( 0 < \omega(t, \lambda_0) < \pi \) in \( (a, b) \) [Coddington–Levinson [22] p.213, 1.15].

Proof. If \( \beta = \pi \), then \( b \) will be the first zero of \( \varphi \). In any case, \( \varphi(t, \lambda_0) \) cannot have zeros in \( (a, b) \). Hence, \( \omega(t, \lambda_0) \) can neither increase to \( \pi \) nor decrease to \( 0 \).

Example 3.58. (Essence: only through reducing a method to its essence may we be able to easily deal with complicated problems)

Isoperimetric problems: The proof of Bendersky [7] p.143, Theorem 27.6] is simpler than that of Fomin–Gelfand [35] p.43, Theorem 1]. This is because the latter proof uses the complicated concept of variational derivative. See Fomin–Gelfand [35] p.28, 1.10; Figure 3]. Although the concept of variational derivative may help understand the circumstance, it does not have much to do with the purpose of calculus of variations. The calculus of variations uses \( y \) as a variable which is the basis of the theory and should not be related further to \( x \). At most, the latter proof is only a special case of the former proof. Holonomic problems: Similarly, the proof Bendersky [7] pp.146–147, Theorem 27.8(i)] is simpler than that of Fomin–Gelfand [35] p.46, Theorem 2]. Non-holonomic problems: Bendersky [7] p.148, 1.10–p.150, 1.5] gives a detailed proof of Fomin–Gelfand [35] p.48, Remark 1]; only through reducing a method to its essence may we be able to easily deal with complicated problems.
Example 3.59. (Networks: why we should emphasize integration studies)

It can be said that physics integrates various topics in differential equations. For example, Jackson [55, §3.11 Expansion of Green Functions in Cylindrical Coordinates] integrates the general form of 3-dim Green’s function [Jackson 55 p.38, (1.31); p.125, (3.142)], the 1-dim Green’s function [Jackson 55 p.125, (3.143)], the Sturm–Liouville system [Jackson 55 p.126, (3.145)] and the Wronskian normalization [Jackson 55 p.126, (3.146)]. In differential equations, we usually treat them as independent and disconnected topics. However, when we put them into one physical system simultaneously to serve a special purpose (for the present case, the computation of the potential of a unit point charge), we should consider and ensure the compatibility among these topics. They are interrelated. The assignment of the value of a parameter of one the present case, the computation of the potential of a unit point charge), we should consider and ensure the compatibility among these topics. They are interrelated. The assignment of the value of a parameter of one

\[ \text{Coddington–Levinson [22, p.192, Theorem 2.2(ii)]}. \]

However, in functional analysis, the Watson [109, p.98, (16)] follows by substituting \( \chi(z) \) into Watson [109, p.286, l.20] instead, our calculation will not obtain the correct electric potential. Except for leading us to the consideration of compatibility, integration studies may also help us

(1). Trace back to natural origins. From the viewpoint of one dimension alone, the formula given in Birkhoff–Rota [10, p.286, l.20] looks artificial. However, once it combines with integration studies, it will become natural: the integration through the normalization of Jackson [55, p.126, (3.146)] reveals that the radial consequence Jackson [55, p.125, (3.143)] originates from the natural symmetric Green’s function in three dimensions [Jackson 55 p.125, (3.142)]

(2). Observe that a side problem for one subject may be the main problem of another subject. Coddington–Levinson [22, p.192, Theorem 2.2(iii)] says that as a function of \( t \), \( G \) satisfies \( Lx = lx \) for \( t \neq \tau \). How about if \( t = \tau \)? Even though the answer may help better understand Coddington–Levinson [22, p.192, Theorem 2.2], we often ignore this side problem. This is because the \( \delta \)-function is an indefinable object in the classical theory of ordinary differential equations. At best, we can only say that \( G^{(n-1)}(\tau, \tau, l) \) does not exist [Coddington–Levinson [22, p.192, Theorem 2.2(ii)]]. However, in functional analysis, the \( \delta \)-function can be rigorously defined [Rudin 87 p.141, l.7–l.15–l.3]. Then the above side problem becomes interesting and can be completely solved [Rudin 87, p.206, Exercise 10; p.378, l.6].

Example 3.60. (Networks: flow charts in design, proof strategies; effectiveness: directness)

Given a task and available resources. By aiming at the goal, we may design a flow chart to complete our mission. Any digressive topic is meaningless for this project.

The general solution of

\[ \frac{d^2y}{dz^2} - \frac{\phi(z)}{\psi(z)} \frac{dy}{dz} + \left( \frac{2}{\psi(z)} \frac{\phi(z)}{\psi(z)} \right)^2 - \frac{3}{\psi(z)} \left( \frac{\phi'(z)}{\psi(z)} \right)^2 + \frac{1}{\psi(z)} \frac{\phi''(z)}{\psi(z)} + \{ \psi^2(z) - \nu^2 + \frac{1}{4} \} \frac{\psi'(z)}{\psi(z)} \right] y = 0 \]

\[ y = \sqrt{\frac{\phi(z)}{\psi(z)}} G(z) \{ \psi(z) \} \] [Watson 109, p.98, l.15–l.37–l.41].

Proof. The proof strategy is to eliminate \( \chi(z) \) in Watson [109, p.98, (12) & (13)].

\[ \chi(z) = \frac{\phi(z)}{\psi(z)} \] [Watson 109, p.98, (14)]

Watson [109, p.98, (16)] follows by substituting \( \chi(z) \{ \psi(z) \}^\nu \phi^{1/2} \) into Watson [109, p.98, (12)].

The strategy for expressing \( \frac{\chi(z)}{\phi(z)} \) in terms of \( \phi(z) \) and \( \psi(z) \):

\[ 2 \frac{\chi(z)}{\phi(z)} = \frac{\phi'(z)}{\phi(z)} - \frac{\psi'(z)}{\psi(z)} - (2\nu - 1) \frac{\psi(z)}{\psi(z)} \] [Watson 109, p.98, l.111].

The strategy for expressing \( \frac{\chi(z)}{\phi(z)} \) in terms of \( \phi(z) \) and \( \psi(z) \):

By differentiating the above equality with respect to \( z \), we have
Remark. We may, but need not, derive formulas for $\chi'(z)$ or $\frac{\chi''(z)}{\chi(z)}$ from the expression for $\chi(z)$.

**Example 3.61.** (Networks: subgroups vs. subfields, theory vs. reality; insightfulness: perspectives [axiomatic approaches vs. heuristic approaches]; avoiding unnecessary complications; essence; accessibility) (Galois theory)

Both Edwards [30] and van der Waerden [102] vol.1, chap.5–chap.7 discuss Galois theory. The latter book uses axiomatization as its guideline to derive its important results. It asks what background is required in order to understand the essence of Galois theory. Then it designs a flow chart in logic that leads to various theorems of the theory. In contrast, Edwards [30] adopts a different approach. First, it explains clearly about the theory’s origin. Second, it asks what tools are needed in order to solve the problem. Then it introduces the concepts of group, subgroup, and normal subgroup, and Galois group. Third, it asks what strategy should be used to solve the problem. The strategy is to use the theorem that a polynomial equation $f(x) = 0$ is solvable by radicals $\iff$ Galois group is solvable (Edwards [30] p.61, l.17–l.19]). The $\Rightarrow$ part provides an effective test (van der Waerden [102] vol.1, p.173, l.14]) for solvability because a finite group is easier to work with than a field. The $\Leftarrow$ part sketches some guidelines about how to find the solutions (van der Waerden [102] vol.1, p.173, l.15]) if the Galois group of the equation is solvable. The key to proving this theorem is contained in Edwards [30] p.58, l.3–l.1]. Fourth, it shows how we use the theory to solve practical problems (Edwards [30] p.133, l.9–p.135, l.4; p.91, l.17–l.19]). The approach given in Edwards [30] enables us to distinguish important theorems (Edwards [30] p.34, l.10–l.11; p.64, Theorem]) from side theorems (Edwards [30] p.72, l.15–l.19; p.82, Exercises 10 & 11]).

**Example 3.62.** (Networks; simplicity; avoiding unnecessary complications; advantages: using group theory to discuss the splitting field of a polynomial) (Definition of Galois groups)

Let $\zeta$ be a primitive $12$th root of unity. The Galois group of $x^{12} - 1 = 0$ over $\mathbb{Q}$ given in Edwards [30] p.51, (1)] considers all the roots of $x^{12} - 1$, i.e., the factorization of $x^{12} - 1$ into linear factors, while the Galois group of $x^{12} - 1 = 0$ over $\mathbb{Q}$ given in van der Waerden [102] vol.1, p.154, l.24] only focuses on the roots of $x^4 - x^2 + 1 = 0$. According to Edwards [30] p.94, Theorem] or van der Waerden [102] vol.1, p.154, l.8], the elements of the Galois group are $\zeta \to \zeta$, $\zeta \to \zeta^5$, $\zeta \to \zeta^7$, $\zeta \to \zeta^{11}$.

If we adopt the terminology of group theory, we can formulate the following theorems more naturally and precisely. Edwards [30] p.56, Exercise 2] states that a subgroup divides a group into right cosets. Edwards [30] p.120, l.22–l.25] states that a Galois subgroup divides a Galois group into right cosets. Edwards [30] p.122, l.6–l.8] states a subgroup divides a group into left cosets. The proof of the first and the third statements essentially use group theory alone, so the proofs are abstract and their resources are limited. In contrast, the proof of the second statement uses a Galois subgroup which corresponds to a specific subfield. The extra resources enrich the meaning of the proof through the interaction between subgroups and subfields. Edwards [30] p.64, l.6–p.65, l.6] uses very awkward language to illustrate the following proposition: $G_i(G_{i+1} \cap G_i) \cong G_{i+1} \cap G_i \cong G_i \cap G_{i+1}$.

It would be much clearer and simpler if we use the theorem given in van der Waerden [102] vol.1, p.141, l.4] instead to prove this proposition.

**Example 3.63.** (Networks: recognizing a theorem’s attributes helps find its proof and determine the role that it plays in a theory)

3.2] are corollaries of Hartman [48, pp.12–13, Theorem 3.1] because it is necessary to prove the existence of a maximum interval before discussing them. In my opinion, the two corollaries and the theorem are corollaries of Hartman [48, p.11, Corollary 2.1] because the key idea of the proofs of the former three is Hartman [48, p.11, Corollary 2.1].

**Example 3.64.** (Networks: a theorem with added features; simplicity)


**Example 3.65.** (Networks: the scope of a method’s applicability)

When a method is related to a problem, we should apply the method to only where it may, and leave the rest to be dealt with in another way. In a finite-dimensional normed space, its various norms are equivalent (Rudin [87, pp.14–15, §1.19]). Consequently, all the properties of finite-dimensional normed spaces remain valid if we replace one norm with another. However, we cannot use this method to prove that Hartman [48, p.26, Lemma 3.2] implies Hartman [48, p.26, Exercise 3.1]. Instead, we should prove the latter statement as follows:

Proof. Let \( h > 0 \).

\[
\lim_{h \to 0} \frac{\|y(t+h)\| - \|y(t)\|}{h} = \frac{dy}{dt} \text{sgn} y(t) (j = 1, \cdots, d) \quad \text{(Hartman [48, p.26, Lemma 3.1]).}
\]

\[
\lim_{h \to 0} \frac{\|y(t+h)\| - \|y(t)\|}{h} = \left( \frac{dy_1}{dt} \text{sgn} y_1(t), \cdots, \frac{dy_d}{dt} \text{sgn} y_d(t) \right).
\]

Taking the Euclidean norm on both sides, we have \(D|y(t)| = |y'(t)|.\)

**Example 3.66.** (Networks: inseparability of a theorem from its role in the entire theory)

We often isolate certain facts from a context and give them the status of a theorem. In doing so, we ignore the inseparability of a theorem from the role it plays in the entire theory.

For the application of a theorem, we would like to study how often it appears in the theory, in which areas it appears, and in what form it fits into its surroundings. For example, let us compare how John [57, p.153, l.8] and Rudin [87, p.180, l.10] introduce the Paley–Wiener theorem into the theory of PDE.

**Example 3.67.** (Networks: relationships)

We may use Green’s functions, integral transforms, or separation of variables to solve PDE’s (Sneddon [92, chap.3]). However, these methods are closely related. More precisely, each method brings out the next one.

We need Green’s function to solve Sneddon [92, p.294, (1),(2),(3)].

\[\text{→ We use the Laplace transform to determine the Green function (Sneddon [92, p.297, l.2]).}\]

\[\text{→ In view of the expansion given in Sneddon [92, p.298, (16)], we may also use separation of variables to determine the Green function (Sneddon [92, p.298, l.6]).}\]

**Example 3.68.** (Networks: going back to the basics to establish the main relationship, unifications)

(1) The lack of development may make us mistake a partial aspect for the big picture.

The way Goldstein uses the inertia tensor may make his readers believe that tensors and linear transformations are the same (Goldstein [40, p.147, l.14–l.29]). In fact, this viewpoint can be justified.
only under certain conditions (Warner [107] p.55, (d)).

Note. Goldstein [40, p.146, (5-9)] is a corollary of Warner [107] p.55, (e).

(2) The relationship between the basic concepts (the tensor algebra and the exterior algebra) can be fully established (Warner [107] p.56, Definition 2.4).

The tensor product and wedge product in Warner [107] pp. 54–65] establish the relationships among three isolated major treatments: Lang’s algebraic treatment (Lang [63, chap. XVI]), Goldstein’s treatment for mechanics (Goldstein [40] p.146, l.15–p.147, l.−8], and Spivak’s superficial treatment for differential geometry (Spivak [95] vol.1, chap. 4 & chap. 7).

(a) The domains of the tensor product and the wedge product are fully developed.

(b) The operations become simple and direct. Example (tensor product): Compare Warner [107] p.54, Definition 2.1] with Spivak [95 vol.1, p.159, l.7].

(c) The artificial outlook of synthetic properties can be illustrated by inner basic operations.

Compare Warner [107] p.56, Definition 2.4; p.57, 2.6(a)] with O’Neill [74 p.153, l.−1].

(3) (Inclusiveness and consistency)

The scheme in Warner [107] pp.54–62, §2.1–§2.13] is consistent with almost every existing concept of product.

(a) Multiplication of real numbers (Warner [107] p.59, l.14)].

(b) Scalar product (Arnold [3, p.173, Problem 7]).

(c) Vector product (Arnold [3, p.173, Problem 6]).

(d) (Lie group with its Lie algebra)

Quaternion product = vector product − scalar product (Pontryagin [82, p.170, l.1]).

Bracket product = vector product (Pontryagin [82, p.384, Example 93]).

(4) (Sorting)

We would like to distinguish between the identification by general properties (the 2nd isomorphism in Warner [107, p.60, l.2]) and the identification by a particular assignment (the 1st isomorphism in Warner [107, p.60, l.2]). Only for the latter may we have the freedom to make a choice for adjustment (Warner [107, p.60, l.5]).

(5) The basics are developed in order to study advanced topics that require clarification. Most common mistakes committed by mathematicians are basics. These basics may look confusing unless they are well-isolated from complicated situations.

(6) The basics are the foundation in building and expanding a theory. They are constantly modified by experimental results in order to advance further research.

Example 3.69. (Networks: quality checklist for a theory of tensors)

(1) Does the theory distinguish a bound vector from a free vector? Good: Kreyszig [61] p.103, l.12–l.16]. Poor: Peebles [78 §8].

(2) Does the theory mention that the allowable coordinate transformations form a group? Good: Kreyszig [61 p.101, l.20–l.21]. Poor: Peebles [78 §8].
Does the theory have a clear definition of a tensor field? Good: Kreyszig [61, p.111, l.12]. Poor: Peebles [78, §8].

Does the theory have a consistent scheme for development? Kreyszig [61, (31.1), (31.2), (31.3) & (32.1)] are proved according to the same scheme, while Peebles [78, p.230, (8.14)] is stipulated by hard and fast rules.

Does the theory have a geometric interpretation for the contravariant or covariant components of a vector? Good: Kreyszig [61, p.116, Fig. 35.1 & Fig. 35.2]. Poor: Peebles [78, §8].

When we use the elements of a vector space as contravariant vectors (Kreyszig [61, p.121, l.-11–l.-10]) and the elements of its dual as covariant vectors (Kreyszig [61, p.123, l.10]) to define tensors, do we relate it to the classical definition with a proper justification? Good: Kreyszig [61, p.122, l.9; p.123, l.-7–l.-6]. Poor: Spivak [95, vol. 1, chap. 4].

Example 3.70. (Networks: how a math network strengthens effectiveness)

(Identification of fundamental groups)

(1) Product: Pontryagin [82, p.350, F].

(2) Homomorphic image: Pontryagin [82, p.370, l.–7]. A covering space is a generalization of a topological group homomorphism (Pontryagin [82, p.134, C]). The use of universal coverings makes it easy to identify the fundamental group of a homomorphic image.

(3) Although we can define the fundamental group in an arcwise connected topological space $R$ (Pontryagin [82, p.348, Definition 44]), for its identification we often wonder where to start if $R$ does not have any group structure.

Remarks. Massey [71] completely ignores the role of topological group homomorphisms when discussing covering spaces. Pontryagin [82] shows the complete development process: Topological group homomorphism $\rightarrow$ covering space $\rightarrow$ covering group. Imposing a group structure on a covering space is like going back to the original stage. Warner [107] lacks the first part of the development process (Topological group homomorphisms $\rightarrow$ covering space).

Example 3.71. (Networks: links among milestones)

The correct approach toward developing a theory is to use the important facts as milestones and then link these facts with theorems. In contrast, the incorrect approach is to use the big theorems as the milestones and link them with examples.

4 Methods of weakening a hypothesis

Suppose we use input-process-output as a model for a method. In some cases, we need to increase the input as shown in the definition of productiveness. In other cases, we need to reduce input. First, a theory is less likely to be contradictory and more likely to be consistent if it contains fewer assumptions.

Example 4.1. (Reducing the input for consistency; hitting multiple targets with one shot)

The four formulas given in Born–Wolf [13, p.41, (19)] are derived from a single figure: Born–Wolf [13, p.39, Fig. 1.10]. In contrast, the two formula given in Jackson [55, p.305, (7.38)] are derived from Jackson [55, p.305, Fig. 7.6(a)]. The two formulas given in Jackson [55, p.306, (7.40)] are derived from Jackson [55, p.305, Fig. 7.6(b)]. It can be said that Born’s method hits two birds with one stone. Born–Wolf [13, p.39, Fig. 1.10] involves only one convention: Born–Wolf [13, p.40, line 12–line 11]. All the calculations in proving Born–Wolf [13, p.41, (19)] follow this convention. Jackson [55, p.305, Fig. 7.6(a)] and Jackson [55, p.305, Fig. 7.6(b)] involve two different conventions: Wannness [106, p.411, line 12–line 11; p.415, line 4–line 5]. More conventions only increase the chance of leading to a contradiction. The explanations given in Wannness [106, p.416, line 12], Jackson [55, p.306, line 7] and Born–Wolf [13, p.42, line 3–line 1] are not satisfactory. In my opinion, for normal incidence, we should consider \( E \) parallel to the plane of incidence because this way fixes the value of \( E \). If we were to consider \( E \) perpendicular to the plane of incidence, it would be difficult to determine whether \( E \) is positive or negative. The dichotomy given in Jackson [55, p.305, Fig. 7.6] unnecessarily uses the same set formulas (Jackson [55, p.304, (7.37)]) twice and fails to produce any extra benefit.

Second, the goal of axiomatization is to minimize a set of axioms that deduce the entire theory. If \( A \Rightarrow B \), then we say that \( A \) is stronger than \( B \) or that \( B \) is weaker than \( A \). Although axioms are strongest statements in a theory, we want to minimize the number of axioms. Third, in order to characterize a concept, we should seek a minimal set of its necessary conditions strong enough to become its sufficient conditions.

Example 4.2. (Reducing the input for characterizing a concept)

In the proof of Perron [79, p.276, Satz 38], Perron shows that the sufficient conditions for convergence originate from its necessary conditions (Perron [79, p.274, line 6–line 4]). This approach enables us to see how the theorem is produced and formulated. In contrast, the proof given in Wall [103, p.37, Theorem 8.1] fails to explain how the sufficient conditions are obtained because it fails to provide a reason for the artificial classification of cases. See Wall [103, p.38, line 4–line 5; line 17].

Fourth, in order to make a theorem stronger, we should weaken its hypothesis while keeping its conclusion the same. The goal of using the method of weakening a hypothesis is to find the weakest hypothesis for a given conclusion. Suppose \( A \) is the hypothesis and \( B \) is the conclusion of a theorem. If \( A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow \cdots \Rightarrow A_n \Rightarrow B \), we want to find \( A_n \), where the \( n \) is the largest. That is, we want to shorten the deduction chain. If Theorem \( A \) and Theorem \( B \) have the same conclusion and the hypothesis of Theorem \( A \) is weaker than that of Theorem \( B \), then we may use a proof of Theorem \( A \) to prove Theorem \( B \) even though it may not be the most effective method to prove Theorem \( B \). We say that the most effective proof of Theorem \( A \) is more refined than the most effective proof of Theorem \( B \) if Theorem \( A \) and Theorem \( B \) have the same conclusion and the hypothesis of Theorem \( A \) is weaker than that of Theorem \( B \). If a hypothesis is modified so that it can be applied to a wider class, then the hypothesis is considered weakened. By weakening the hypothesis of a theorem, we may pinpoint the exact reason that leads to the conclusion. In the rest of this section, we will discuss methods of weakening the hypothesis alone.
4.1 Examples of weakening the hypothesis of a theorem while keeping its conclusion the same

In the following chains, the hypothesis of each theorem is weaker than that of the previous theorem:

(1) Zygmund [113, vol.1, p.78, l.7–l.6] → Zygmund [113, vol.1, p.78, Theorem 1.26]
Hypothesis: \( (u_V = o(1/V)) \) \( \rightarrow \) \( (u_V = O(1/V)) \)

(2) Zygmund [113, vol.1, p.81, Theorem 1.36] → Zygmund [113, vol.1, p.81, Theorem 1.38]
Hypothesis: \( (u_n = o(1/n)) \) \( \rightarrow \) \( (u_n = O(1/n)) \)

(3) Zygmund [113, vol.1, p.89, Theorem 3.4] → Zygmund [113, vol.1, p.90, Theorem 3.9]
Hypothesis: \( (x_0 \text{ is a point of continuity of } f) \) \( \rightarrow \) \( \[ \Phi_{x_0}(h) = o(h) \] \) (Zygmund [113, vol.1, p.50, l.13; p.65, l.12])

(4) (Cahcuy’s integral theorem: Conway [24, p.73, Proposition 2.15])(a consequence of Green’s theorem) → Rudin [88, p.221, Theorem 10.13]
Hypothesis: analyticity (Conway [24, p.34, Definition 2.3]) \( \rightarrow \) differentiability (Conway [24, p.96, Goursats Theorem])

(5) The Stone-Weierstrass theorem: Rudin [86, p.146, Theorem 7.24](the real case [respectively, the complex case]) \( \rightarrow \) Rudin [86, p.150, Theorem 7.30] [respectively, Rudin [86, p.152, Theorem 7.31]]
Hypothesis: \( (.\mathcal{A} \text{ is the algebra of real [respectively, complex] polynomials}) \) \( \rightarrow \) \( (.\mathcal{A} \text{ satisfies the hypothesis of Rudin [86, p.150, Theorem 7.30][respectively, Rudin [86, p.152, Theorem 7.31]])} \)

(6) Uniqueness theorems about generalized Lipschitz conditions: Coddington–Levinson [22, p.10, Theorem 2.2] \( \rightarrow \) Coddington–Levinson [22, pp.48–49, Theorem 2.1] \( \rightarrow \) Coddington–Levinson [22, p.49, Theorem 2.2] (respectively, Coddington–Levinson [22, p.51, Theorem 2.3])

The hypothesis of Coddington–Levinson [22, pp.48–49, Theorem 2.1] is weaker than that of Coddington–Levinson [22, p.10, Theorem 2.2] (see Coddington–Levinson [22, p.49, l.12–l.19]). The hypothesis of Coddington–Levinson [22, p.49, Theorem 2.2] (respectively, Coddington–Levinson [22, p.51, Theorem 2.3]) is weaker than that of Coddington–Levinson [22, pp.48–49, Theorem 2.1] (see Coddington–Levinson [22, p.49, l.20] (respectively, Coddington–Levinson [22, p.51, l.12–l.19])

Remark 1. There can be following two versions of Zygmund [113, vol.1, p.90, Theorem 3.9]:

A \( \sigma_n(x) \rightarrow f(x) \) for every \( x \) satisfying \( \Phi_x(h) = o(h) \).

B \( \sigma_n(x) \rightarrow f(x) \) almost everywhere.

If we adopt version [1A] we may use it to prove Zygmund [113, vol.1, p.89, Theorem 3.4]. In contrast, if we adopt version [1B] we will reach a point of no return. Namely, we can no longer use version [1B] to prove Zygmund [113, vol.1, p.89, Theorem 3.4]. This is because the existence of \( x \) in version [1A] is constructive (more specifically, \( x \) is fixed), while the existence of \( x \) in version [1B] is less effective because it is derived from reduction to absurdity. Modern mathematicians love to use the term “almost everywhere” in real analysis simply because the meaning of this term is easier to remember than the meaning of \( \Phi_x(h) \). This is the reason why delicate methods of weakening a hypothesis have almost become endangered species in real analysis.
Remark 2. Let $T = \{ z \in \mathbb{C} | |z| = 1 \}$, $e^{i \theta_0} \in T$ and $U \in L^1(T)$. (If $U \in C(T)$, then $P_U(z)$ [Ahlfors [11] p.167, l. 1-14] is continuous on $\{ z \in \mathbb{C} | |z| \leq 1 \}$ $\rightarrow$ [If $U$ is continuous at $z = e^{i \theta_0}$, then $\lim_{z \to e^{i \theta_0}} P_U(z) = U(e^{i \theta_0})$ (Ahlfors [11] p.168, Theorem 25)]. See Ahlfors [11] p.167, l.9-l.12] for motivation. In contrast, the use of the phrase “almost everywhere” in Rudin [88, p.258, Corollary] prevents us from knowing the exact locations of $z = e^{i \theta}$ at which the formula given in Rudin [88, p.258, l.4] is valid.

4.2 How we recognize and appreciate the value of methods of weakening a hypothesis

In order to fully understand a method of weakening the hypothesis, we should not only know what it is, but also recognize its value and key points.

(1) We want to know from where the method comes. What problems motivate mathematicians to create such a device? What obstacle does this method of weakening a hypothesis can conquer, while other old methods cannot?

Suppose $z = \infty$ is a singularity of the second kind, we know the solutions of Coddington–Levinson [22, p.151, (4.1)] for the real case, and we want to find the solutions for the complex case (Coddington–Levinson [22, p.161, l. 8-1.l.5]). Then it requires to replace the boundedness of $f$ at $z = \infty$ in Conway [24, p.125, Theorem 1.4] with a growth condition, i.e., to prove Conway [24, p.135, Corollary 4.2]. See Conway [24, p.124, l. 2-p.125, l.1] and Coddington–Levinson [22, p.164, l.1-10].

(2) We should not take a musket to kill a butterfly

We should highlight the amazing effects that a refined method of weakening the hypothesis produces. If an old, crude method can do, it is unnecessary to use a new, refined method of weakening the hypothesis. Using refined methods to do crude things is a unnecessary waste. For example, it is unnecessary to use the Phragmen–Lindelöf method to prove Rudin [88, p.274, Theorem 12.8]: we can prove the statement given in Rudin [88, p.275, l.11] using Conway [24, p.125, Theorem 1.4]. To specify a bound given the boundedness of $f$ at $z = \infty$ is not as amazing as to specify a bound given the growth condition of $f$ because the condition of the latter statement is weakened. Compare Rudin [88, p.274, Theorem 12.8] with Conway [24, p.135, Corollary 4.2].

(3) How to highlight the key idea of a method of weakening the hypothesis

(a) Use the method of standardization to eliminate unnecessary complications. For example, use a symmetric case (Conway [24, p.135, Corollary 4.2]) to represent the general case (Coddington–Levinson [22, p.162, Theorem A]) without loss of generality. See Conway [24, p.135, l. 3-l.1].

(b) For the formulation of a method of weakening the hypothesis, we should trace the method’s origin and preserve its original setting. For example, Conway [24, p.135, Corollary 4.2] is a right version; see Conway [24, p.124, 1-p.125, l.1]. Adopting other versions such as Conway [24, pp.134–135, Theorem 4.1] or Rudin [88, p.276, Theorem 12.9] may distract us from the essence of the Pragmen–Lindelöf method.
5 Physical methods

5.1 Physical interpretations of a problem

Consider Laplace’s equation (Watson–Whittaker [108, p.386, (1)] and Born–Wolf [13, p.11, (7)]). When we select a coordinate system, we should choose one suitable for the geometric symmetry of the shape of object (Jackson [55, p.104, l.6–l.12]).

5.2 Physical interpretations of a solution

Consider the solutions given by Born–Wolf [13, p.16, (8)] and Watson–Whittaker [108, p.397, l.8–l.19].

1. Physical considerations help select meaningful solutions (Jackson [55, p.107, l.–l.–1] and Cohen-Tannoudji–Diu–Laloë [23, p.648, (C-9); p.652, l.17, p.664, l.2]).

2. Solutions must be well-defined: In Jackson [55, p.104, l.15], we consider \( x = \cos \theta \) instead of \( \theta \); in Jackson [55, p.105, l.–5], we restrict \( r \) to be greater than 0.

3. Physical considerations help select an appropriate solution form (Jackson [55, p.104, l.–6–l.–5]).


2. From the viewpoint of individual particles using the Newtonian mechanics: The results are summarized in Marion–Thornton [70, p.487, Table 12-1]; the pictorial features are given in Marion–Thornton [70, p.472, Fig. 12-2].

3. From the viewpoint of the entire system using the Lagrangian in the Lagrangian mechanics:
   
   (a) If the equations connecting the generalized coordinates and the rectangular coordinates do not explicitly contain the time, then the kinetic energy has the form given in Marion–Thornton [70, p.476, (12.18)].

   (b) The expansion of the potential energy in a Taylor series about the equilibrium configuration yields Marion–Thornton [70, p.476, (12.32)].

   (c) The Lagrangian equations yield Marion–Thornton [70, p.478, (12.38)]. By substituting Marion–Thornton [70, p.478, (12.39)] into Marion–Thornton [70, p.478, (12.38)], we have Marion–Thornton [70, p.479, (12.40)]. In order to find the solutions of Marion–Thornton [70, p.479, (12.40)], we solve \( \omega \) for Marion–Thornton [70, p.479, (12.42)] first. Then for each \( \omega \), we solve Marion–Thornton [70, p.479, (12.40)] to obtain the corresponding eigenvector \( a_r \).

   (d) Using Marion–Thornton [70, p.483, (12.63)], we simultaneously diagonalize \( T \) and \( U \) [Marion–Thornton [70, p.484, (12.65); (12.66)]]. Then the Lagrangian equations in normal coordinates become completely separable [Marion–Thornton [70, p.485, l.4]].
(4) From the viewpoint of the entire system using the Hamiltonian operator in quantum mechanics: The first equality given in Marion–Thornton [70, p.480, (12.45)] is a special case of Cohen-Tannoudji–Diu–Laloé [23, vol.1, p.576, (4)]. By Cohen-Tannoudji–Diu–Laloé [23, vol.1, pp.584–585, Complement $H_V$, 2d]], we find that $<X_G>(t)$ and $<X_R>(t)$ oscillate at angular frequencies of $\omega_G$ and $\omega_R$, which agrees with the classical result.

Remark. As we go to a more advanced level and widen our consideration, new physical meanings of mathematical equations continue to develop and meanings of equations become richer and more delicate. Nonetheless the meanings in older theories are still well-preserved in a newer theory.

5.3 How we understand the physical meaning of a mathematical theorem

In view of Jackson [55, p.36, l.14–p.37, l.16; p.37, l.15–l.14; p.38, l.11–l.17; p.39, (1.42)–(1.46)], the concept of dipole layer is the key to understanding the physical meaning of Green’s theorem or those of boundary conditions. This is the reason why Jackson discusses dipole layers [Jackson [55, §1.6]] before boundary conditions [Jackson [55, §1.8–§1.10]]. However, it is difficult to understand the former topic without knowing dipoles or point dipoles in advance. Therefore, it would be better prepared for understanding if one read Jackson [55, §1.6] again after being familiar with dipoles and point dipoles. Remark. By Jackson [55, p.35, (1.31)] and Wangness [106, p.36, (1-135)],

$|\nabla^2 (\frac{1}{|x-x'|})| = -4\pi \delta(x-x')$ [Jackson [55, p.36, l.3]].

5.4 Physical ideas vs. their formal formulations

Physical ideas are usually simple, but their formal formulations in mathematical language can be sophisticated.

Example 5.2. (Watson’s lemma)

Watson’s lemma considers the integral $\int_0^\infty e^{-zr}f(t)dt$. The dominant value of the integral occurs near $t = 0$. This observation suggests that we estimate the integral by replacing $f$ with its local expansion at $t = 0$. For the formal formulation of the lemma, see Koekoek [60, Theorem 2].

5.5 Physical proofs

5.5.1 A theorem’s proof should be guided by its physical theme

Guided by a theorem’s physical theme, one may develop a better strategy to prove it.

Example 5.3.

Coddington–Levinson [22, p.319, l.15–l.10] uses the following argument: if $[(A \land C) \Rightarrow B]$ and $(B \Rightarrow C)$, then $(A \Rightarrow B)$ ($\ast$), where

$A = $ Coddington–Levinson [22, p.318, (1.16)] & (1.17)];

$B = $ Coddington–Levinson [22, p.319, (1.23)] (see Coddington–Levinson [22, p.319, l.15–l.16]);

$C = (|\varphi(t)| \leq \delta$ and Coddington–Levinson [22, p.319, (1.22)]) (see Coddington–Levinson [22, p.319, l.15–l.16]);

$\ast$ Coddington–Levinson [22, p.319, (1.23)].
If in (+) we substitute \(C\) into \(B\), we see that the conclusion \((A \Rightarrow C)\) is false. Thus Levinson’s argument is incorrect. However, the hypothesis \(((A \land C) \Rightarrow B)\) and \((B \Rightarrow C)\) ensures that if \(A\) holds, then \(B\) and \(C\) are equivalent. We can correct Levinson’s mistake by the following method:

Even though the estimate provided by Pontryagin \([81, p.211, l.11]\) is poorer than that given in Coddington–Levinson \([22, p.319, (1.23)]\), we may use the former estimate to prove \(C\). Thereby, we can obtain the better estimate \(B\).

**Example 5.4.** (The mathematical formulation of the second law of thermodynamics leads to a criterion for integrability of Pfaffian forms)\([\text{Sneddon} [92, p.41, l.16–l.27; p.35, Theorem 8]; \text{Zemansky–Dittman} [112, p.169, (7-7); p.170, l.10–l.12; p.173, l.4–l.111]\]

The algebraic criterion for integrability of Pfaffian forms \([\text{Sneddon} [92, p.21, Theorem 5]]\) is good for calculation, while the geometric (or physical) criterion for integrability of Pfaffian forms \([\text{Sneddon} [92, p.34, Theorem 7; p.35, Theorem 8]]\) is good for geometric (or physical) considerations. Pfaffian forms and thermodynamics are closely related. Without considering thermodynamics we cannot see the insight of Pfaffian forms; Without considering Pfaffian forms, we would have no mathematical foundation for thermodynamics. One should establish a solid connection between the two fields:

The connection from Pfaffian forms to thermodynamics: By Zemansky–Dittman \([112, p.173, (7-13); p.174, (7-14)]\), the function \(\mu\) is, apart from a multiplicative constant, a function only of the empirical temperature of the system \([\text{Sneddon} [92, p.41, l.1–l.4–l.111]]\). The connection from thermodynamics to Pfaffian forms: By Sneddon \([92, p.41, l.1–l.16; l.1–l.17; l.1–l.27; p.35, Theorem 5; p.34, Theorem 7; p.35, Theorem 8]]\) is good for geometric (or physical) considerations. Pfaffian forms and thermodynamics are closely related. Without considering thermodynamics we cannot see the insight of Pfaffian forms; Without considering Pfaffian forms, we would have no mathematical foundation for thermodynamics. One should establish a solid connection between the two fields:

The differences between thermodynamics and the general theory of Pfaffian forms: Zemansky–Dittman \([112, p.173, l.8–l.10; l.14–l.16]]\). The differences between thermodynamics and the general theory of Pfaffian forms: Zemansky–Dittman \([112, p.173, l.8–l.10; l.14–l.16]]\).

**Example 5.5.** (A physical proof may lay bare the key idea with one penetrating remark: Faraday’s law in moving media)

If we use vector calculus alone to prove Faraday’s law in moving media \([\text{Choudhury} [20, \S 6.3]]\), we may easily miss important physical meanings. Studying is like making friends with someone. We want to know not only him but also his friends because his friends are an indispensable part of him. Sometimes, a mathematical proof requires advanced and complicated knowledge and a long argument; we may easily get lost. In contrast, a physical proof may often lay bare the key idea with one penetrating remark.

Both Wangsness \([106, p.269, l.1–l.4–p.272, l.10]\) and Choudhury \([20, p.250, l.1–l.252, l.4–l.14]\) prove Faraday’s law in moving media. The formula given in Wangsness \([106, p.271, (17-25)]\) is the one given in Choudhury \([20, p.251, (6.18)]\), both proofs after this formula are the same. Both proofs before this formula attempt to prove this formula, but they do with different approaches. The latter uses vector calculus; we may learn the calculation of surface integrals \([\text{Choudhury} [20, p.250, l.1–l.10]; \text{Choudhury} [20, p.577, (I.31)]\] \([\text{Choudhury} [20, p.251, l.19]]\). In contrast, the former uses only the definition of derivative; we may miss a lot if we follow the former approach alone.

Both Wangsness \([106, \S 17-3]\) and Choudhury \([20, \S 6.3]\) prove that Wangsness \([106, p.264, (17-3)]\) implies Wangsness \([106, p.272, (17-30)]\). The proof of Faraday’s law in \([https://en.wikipedia.org/wiki/Faraday%27s_law_of_induction](https://en.wikipedia.org/wiki/Faraday%27s_law_of_induction)\) proves the converse (Wangsness \([106, p.272, (17-30)]\) implies Wangsness \([106, p.264, (17-3)]\)) and indicates that their equivalence is due to the interchangeability between \(v_l\) [\(l\) stands for “loop”] and \(v_m\).
understand the essence of Carathéodory’s proof if one fail to know its physical meaning. In contrast, Born’s proof is closely related to the measurement of entropy using a quasi-static process. It would be difficult to Remark. Continuously deforming the cylinder [Sneddon [92, p.38, l.14–l.19]] refers to reducing the cross section area of \( \sigma \) to 0. The band of accessible points [Sneddon [92, p.38, l.1–13–l.12]] refers to the segment \( IG_0 \).

5.5.2 A physical proof is usually more direct than a geometric proof

**Example 5.6.** (The geometric criterion for integrability of the Pfaffian differential equation)

For Sneddon [92, p.35, Theorem 8], Carathéodory’s thermodynamic proof is more direct than Born’s geometric proof because the latter proof uses reduction to absurdity in Sneddon [92, p.38, l.1–16–l.9]. From the similarity between the path given in Sneddon [92, p.36, Fig. 11] and the integral path of Reif [84, p.160, (5-4-2)], we see that Carathéodory’s idea originates from solving Reif [84, p.160, (5-4-1)]. His proof is closely related to the measurement of entropy using a quasi-static process. It would be difficult to understand the essence of Carathéodory’s proof if one fail to know its physical meaning. In contrast, Born’s proof involves only the geometric shape of solutions of the Pfaffian differential equation. One cannot use Born’s method to measure entropies.

**Remark.** Continuously deforming the cylinder [Sneddon [92, p.38, l.1–15–l.14]] refers to reducing the cross section area of \( \sigma \) to 0. The band of accessible points [Sneddon [92, p.38, l.1–13–l.12]] refers to the segment \( IG_0 \).

5.5.3 Physical proofs are better than analytic proofs

Color painting adds more dimensions and varieties to black-and-white drawing. Similarly, physical and geometric proofs provide more meanings, pictures, insights, and interesting stories than analytic proofs.

In terms of the publishing dates of the above textbooks, the proofs of the later published books are better. The improvements are as follows:

(1) The choices of notations, coordinate systems, orthonormal functions become more compatible to the physical theme of the theorem.


(b) Orthonormal functions: Since we are discussing the solutions of Laplace’s equation in spherical coordinates (Jackson [55, p.95, l.1–13] and Watson–Whittaker [108, p.391, l.1–14]), it is more appropriate to choose \( Y_{lm} \) on the unit sphere instead of \( P_{mn} \) on \([-1, 1]\) as the desired set of orthonormal functions (Jackson [55, p.108, l.16]).

(2) Ideally, the best physical proof is the one each of whose step has a pertinent physical interpretation. The development of physical methods shows the tendency toward such an ideal:

(a) The choice of \( n \) given in Watson–Whittaker [108, p.395, l.12] lacks physical motivation, while the proof of Cohen-Tannoudji–Diu–Laloë [23, vol.1, p.688, (72)] supplies a physical reason: An eigenfunction of the angular momentum \( L^2 \) remains as an eigenfunction with the same eigenvalue after a rotation. The fact that the rotation operators commute with \( L^2 \) (Cohen-Tannoudji–Diu–Laloë [23, vol.1, p.688, l.15–l.14; p.699, (57)]) is more obvious than the fact that \( \nabla^2 \) is invariant under the rotation operators (Jackson [55, p.110, l.8]).

(b) Strictly speaking, Watson leaves a gap in the proof of the formula given in Watson–Whittaker [108, p.395, l.15]. Because \( \theta'_1 \) is a function of \( (\theta, \phi) \) and \( (\theta', \phi') \), he should have expressed \( P_n(\cos \theta'_1) \) as an expansion of spherical harmonics in a form similar to that of the formula given in Cohen-Tannoudji–Diu–Laloë [23, vol.1, p.688, (74)]. If the expansion involved a term \( Y_{km} \), where \( k \) is other than \( n \), then he would not be able to derive the formula given in Watson–Whittaker [108, p.395, l.15]. Either poor notations or the lack of physical motivations fails him to detect the said gap.


(d) If we correct the above shortcomings and make the following changes, the proof given in Jackson [55, §3.6] would be perfect.

\[
[4\pi(2l+1)^{-1}]^{1/2}Y^*_{lm}(\theta(\gamma, \beta), \phi(\gamma, \beta)) = \sum_{m=-l}^{l} A_{lm} Y_{lm}(\gamma, \beta) \quad (\text{Jackson} [55, p.109, (3.58)] \text{ and} \text{Cohen-Tannoudji–Diu–Laloë [23, vol.1, p.688, l.12]}).
\]

Let \( \gamma = 0 \). We have \( [4\pi(2l+1)^{-1}]^{1/2}Y^*_{lm}(\theta', \phi') = A_{l0} = A_m(\theta', \phi') \) (Jackson [55, p.366, (3.66) and (3.60)]).
5.6 A proper physics model can be a natural guide to the study of PDEs

In order to complete the study of wave equations, we must consider the following three cases. For each case, we need to choose a proper physics model as a guide.

Case 1. 1 dimensional case, rectangular coordinates: continuous strings [Marion [70 §13.4–§13.8]].

Case 2. 2 or 3 dimensional case, polar or cylindrical coordinates: circular membranes [Asmar [5] §4.2–§4.3].

Case 3. 3 dimensional case, spherical coordinates: electric potentials [Jackson [55] §3.1–§3.6].

Each case lays the basis for studying the next one. The proper model is a template for all other similar models. Physics models and solutions of PDEs are inseparable and complement each other. Without a physics model as a guide, PDEs become dull and abstract. Only through a model may we propose significant questions and effectively find their solutions. Without choosing significant boundary and initial conditions, the boundary value problems can become practically meaningless. Although electric potentials can also be used as a physics model for Case 2 [Jackson [55] §3.7–§3.8], they are not as good as circular membranes because we cannot see the former.

6 Improvements of classical methods

If there are drawbacks in a classical method, all we have to do is provide ideas to improve them. If there is a gap in its proof, we simply fill the gap. In other words, a remedy rather than a thorough revamp is all we need. This introduction mode based on needs may make the key to improvement most outstanding.

Example 6.1. (Precision improvement of a classical method)

Let $S$ be a ruled surface [Bell [6, p.313, l.1–l.1]]. If $\alpha'b' - \beta'a' = 0$ [Bell [6, p.314, l.15–l.16]], then $S$ is developable [Bell [6, p.314, l.15–l.16]].

Proof. We have $d = O(\delta t^3)$ in Bell [6, p.314, l.12–l.13], but Bell jumps to the conclusion that $d = 0$. Thus, there is a gap needed to be filled.

Let the directrix of $S$ be $y(s) = (\alpha(s), \beta(s), 0)$, $z(s) = (a(s), b(s), 1)$, and $x(s, t) = y(s) + tz(s)$.

$S$ is developable

\[0 = |y'z'| = \begin{vmatrix} \alpha' & \beta' & 0 \\ a & b & 1 \\ d' & b' & 0 \end{vmatrix} = \beta'a' - \alpha'b' \] [Kreyszig [61] p.169, Theorem 59.1].

Remark. In the above proof, we use the method of differential geometry to fill the gap of a classical proof. Thus, we see the advantage of modern geometry. At the same time, we also see the concept of “consecutive” generators is useful to the intuitive understanding of a ruled surface although it is difficult to make its definition rigorous. Consequently, classical geometry and differential geometry are complementary to each other.

In analytic geometry, we discuss geometry with coordinate systems. Geometry is our main study goal and coordinate systems are nothing but tools to express geometric objects as equations. We should choose the coordinate system that makes the equation of the main geometric object in the simplest form. This approach
will allow us to reduce calculations, to easily recognize its properties, etc. For example, when discussing plane sections of a conicoid, we should express these conics in standard form instead of general form.

**Example 6.2.** (Standard form vs. general form)

All parallel plane sections of a conicoid are similar and similarly situated conics [Bell [6] p.74, Ex. 3].

**Proof 1.** Bell [6] p.74, l.12–l.16
By Fine–Thompson [33] p.137, l.1–l.16, the centers of resulting conics are collinear and vary with \(a, h, g, b, f\).

By Fine–Thompson [33] p.137, l.11, the axes of a resulting conic is determined by \(\lambda\), which is, in turn, determined by \(a, b, h\). Thus, the axes of every resulting conic make the same angles with plane coordinate axes, so the conics are similarly situated.

By Fine–Thompson [33] p.138, l.17, the conics are similar.

Remark. In the first proof, we express a plane in simple form \(z = k\); in the second proof, we express a plane in general form \(lx + my + nz = p\). In the first proof, our goal is to find the standard form of a conic, and then its axes and direction-cosines of the axes. The goal leads directly to solutions; the approach helps us see the insight and key ideas. The second proof relies on the comparison between Bell [6 §86, (2) & (3)] and Bell [6 §87, (2) & (3)]. These formulas are derived from the necessary and sufficient conditions for a plane to touch a cone given in Bell [6 p.120, l.1–l.4]. Consequently, the second proof is not as simple and direct as the first one. This example shows that simplification and standardization are the keys to effective studying analytic geometry.

**Example 6.3.** (Simplifying complicity: one right coordinate system does it all)

(Intersection of three planes) Bell [6 §45]
Consider the system of equations given in Bell [6 p.49, (1), (2), & (3)]. Let

\[
\begin{align*}
\mathbf{A} & = \begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
\end{pmatrix}, \\
\mathbf{A}' & = \begin{pmatrix}
a_1 & b_1 & c_1 & d_1 \\
a_2 & b_2 & c_2 & d_2 \\
a_3 & b_3 & c_3 & d_3 \\
\end{pmatrix}.
\end{align*}
\]

\(r = \text{rank of the coefficient matrix } \mathbf{A}\) and \(r' = \text{rank of the argumented matrix } \mathbf{A}'\).
Table 1: The intersection of three planes

<table>
<thead>
<tr>
<th>Systems</th>
<th>Case number</th>
<th>Algebraic Classification</th>
<th>Geometric classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consistent</td>
<td>1</td>
<td>( r = 3 )</td>
<td>Three planes intersect at one point.</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>( r = r' = 2 ); no two rows of the argumented matrix are proportional.</td>
<td>Three planes intersect in one line.</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>( r = r' = 2 ); two rows of the argumented matrix are proportional.</td>
<td>Two planes are coincident, and the third cuts the others.</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>( r = r' = 1 )</td>
<td>All three planes are coincident.</td>
</tr>
<tr>
<td>Inconsistent</td>
<td>5</td>
<td>( r = 2, r' = 3 ); no two rows of the coefficient matrix are proportional.</td>
<td>Normals are coplanar, planes intersect in pairs, and the intersecting lines form a triangular prism.</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>( r = 2, r' = 3 ); two rows of the coefficient matrix are proportional, but the same two rows of the argumented matrix are not proportional.</td>
<td>Two parallel planes intersect a third plane.</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>( r = 1, r' = 2 ); no two rows of the argumented matrix are proportional.</td>
<td>All planes are parallel and distinct.</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>( r = 1, r' = 2 ); two rows of the argumented matrix are proportional.</td>
<td>Two planes are coincident, and the third is parallel.</td>
</tr>
</tbody>
</table>

**Proof.** Based on geometric considerations, there are no cases other than the above eight cases. In order to prove that the two corresponding classifications are equivalent, all we have to do is find the coordinate system to put a case in simplest equation form and then determine the ranks. Because ranks are invariant under translations and rotations and the general case can be obtained from a simple case by a finite number of translations and rotations, it is unnecessary to consider the general case. For example, for case 5, all we have to do is consider \( y = ax, y = \alpha x, y = \beta x \), where \( a \neq 0 \) and \( \alpha \neq \beta \).

**Remark.** Bell [6, §45] attempts to prove the same thing, but it chooses the hard way. In this context, the emphasis should be on geometry rather than matrix theory. Bell might know some matrix theory, but he failed to master it or make good use of it.

**Example 6.4.**

Fine–Thompson [33] pp.60–61, §79 A–C give three derivations of equation of tangent. In fact, they are all derived from the viewpoint of calculus: the tangent line is the limit of secant lines. Thus, the three derivations are just three ways of constructing a secant line of a conic. From the viewpoint of differential geometry, in order to find the tangent line, we would find the normal first [O’Neill [74] p.127, Theorem 1.4; p.148, Lemma 3.8] because in three dimensions the normal determines the tangent plane. The differential-geometric approach is more direct.

**Remark.** The same discussion applies to Fine–Thompson [33] pp.81–83, §102 A–C.

**Example 6.5.** (Corresponding versions of the same idea motivate us to find the general case)
(a) Polars versus polar planes

(i) 2-dim: The polar of a point with respect to a conic [Fine–Thompson [33, p.148, l.16]]

(ii) 3-dim: The polar plane of a point with respect to a conicoid [Bell [6, p.104, l.6–l.5]]

(b) Symmetry between two poles [resp. polar lines]

(i) 2-dim: The polar of $P_1$ passes through $P_2$ [Fine–Thompson [33, p.148, l.7–l.6]]

(ii) 3-dim: The polar plane of $(\alpha, \beta, \gamma)$ passes through $(\xi, \eta, \zeta)$ [Bell [6, p.105, l.12–l.10]]; the polar plane of any point on a line $AB$ passes through a line $PQ$ [Bell [6, p.105, l.10–l.5]]

The general case: all the above concepts or statements can be generalized to n-dimensional manifolds.

**Example 6.6.** (Integration of algebra and geometry)

Fine–Thompson [33, §274] provides an algebraic derivation of the equation for the plane through three given points. It seems that Fine talks shop all the time. Actually, we may also give a geometric derivation.

The direction-cosines of the plane’s normal are proportional to $(x_2 - x_1, y_2 - y_1, z_2 - z_1) \times (x_3 - x_1, y_3 - y_1, z_3 - z_1)$.

Consequently, the equation for the plane is

$$(x - x_1, y - y_1, z - z_1) \times (x_2 - x_1, y_2 - y_1, z_2 - z_1) \cdot (x - x_1, y - y_1, z - z_1) = 0.$$ Namely,$$\begin{vmatrix}
x - x_1 & y - y_1 & z - z_1 \\
x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\
x_3 - x_1 & y_3 - y_1 & z_3 - z_1
\end{vmatrix} = 0 \text{ [Kreyszig [61, p.17, (5.14)].]}
$$

Remark. Note that

$$\begin{vmatrix}
x - x_1 & y - y_1 & z - z_1 \\
x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\
x_3 - x_1 & y_3 - y_1 & z_3 - z_1
\end{vmatrix} = \begin{vmatrix}
x - x_1 & y - y_1 & z - z_1 & 0 \\
x_1 & y_1 & z_1 & 1
\end{vmatrix} = \begin{vmatrix}
x & y & z & 1 \\
x_1 & y_1 & z_1 & 1
\end{vmatrix}, \begin{vmatrix}
x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\
x_2 & y_2 & z_2 & 1
\end{vmatrix}, \begin{vmatrix}
x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \\
x_3 & y_3 & z_3 & 1
\end{vmatrix}.$$ These equalities integrate the algebraic and geometric meanings of the equation for the plane and explain the reason why the proof in Fine–Thompson [33, §276A] is equivalent to that in Fine–Thompson [33, §276C]. The above geometric derivation makes the proof of the statement given in Fine–Thompson [33, p.209, l.11–l.12] become easy.

**Example 6.7.** (Cartesian coordinates lack ability to distinguish infinities of all directions)

The center is $(\infty, \infty, \infty)$ [Fine–Thompson [33, p.285, l.7]].

**Proof.** The algebraic proof follows from Fine–Thompson [33, p.266, (8)]. A geometric proof proceeds as follows: Let the center of conicoid be $(x_0, y_0, z_0)$. Since $(x_0, y_0, z_0)$ is the common midpoint of chords of the parabola on the plane $y = 0$ [Fine–Thompson [33, p.243, Figure]], $y = 0, x_0 = \infty = z_0$. The line through $(\infty, 0, \infty)$ with direction cosines $(\lambda, 0, v)$ is $\frac{x - \infty}{\lambda} = \frac{y - 0}{0} = \frac{z - \infty}{v}$, which can also be interpreted as $\frac{x - \infty}{\lambda} = \frac{y - \infty}{v} = \frac{z - \infty}{v}$, the line through $(\infty, \infty, \infty)$ with direction cosines $(\lambda, 0, v)$. This is why we find from the algebraic proof that the distance between $(\infty, \infty, \infty)$ and any point on the paraboloid is the same. Consequently, from the algebraic viewpoint, $(\infty, 0, \infty) = (\infty, \infty, \infty)$. Since $(x_0, y_0, z_0)$ is the common midpoint of chords of the parabola on the plane $x = 0$ [Fine–Thompson [33, p.243, Figure]], $x = 0, y_0 = \infty = z_0$. 

Remark. The paraboloid has center at $(\infty, \infty, \infty)$ because we use an improper tool. Cartesian coordinates lack ability to distinguish infinities of all directions. If we use spherical coordinates instead, then $(\theta, \phi, r = \infty)$’s will represent different points if $(\theta, \phi)$’s point to different directions. In this case, there will be no common midpoint for the chords through the origin.
Example 6.8. (If the set of centers is empty and we allow a point involving ∞ to be its element due to a tool abuse, then all the theorems to which the false existence of elements leads will be meaningless [Bell [6, §152–§153]])

Consider the system of equations given in Bell [6, p.216, (1), (2), & (3)]. Let

\[ r = \text{rank of the coefficient matrix } \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \]

and

\[ r' = \text{rank of the augmented matrix } \begin{pmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \end{pmatrix} \]

The set of centers may be any of the following cases:

(i) A point: \( r = 3 \) [Table 1], (ellipsoid, hyperboloid, or cone).
(ii) A line: \( r = r' = 2 \); no two rows of the argumented matrix are proportional (elliptic or hyperbolic, cylinder, pair of intersecting planes).
(iii) A plane: \( r = r' = 1 \) (pair of parallel planes).
(iv) The empty set \( \emptyset \).

The classification of cases for centers should be determined by the final solution of a system of equations rather than the solution process. Consider the parabola \( x^2 = 4ay \) with polar coordinates. Let \( A = (\cos \pi/4, \sin \pi/4), B = (\cos \pi/3, \sin \pi/3) \). The middle point for the chord along \( OA = (\infty, \pi/4) \neq (\infty, \pi/3) \) = the middle point for the chord along \( OB \). We may find the algebraic solution \( (\infty, \infty) \) for the center [Example 6.7] because the Cartesian coordinates are inadequate to tell the above difference. In this example, the equation for the second central plane is \( 4a = 0 \), which is the empty set instead of a plane. Thus, this standard type becomes an exception for the classification given in Bell [6, p.216, l.−2–p.217, l.16]. If the set of centers is empty and we allow a point involving \( \infty \) to be its element due to a tool abuse, then all the theorems to which the false existence of elements leads will be meaningless.

On the one hand, a general theory without giving examples lacks concrete pictures. On the other hand, if we focus on examples alone or the description of examples fails to be consistent with the direction of the general theory’s development, then we may not be able to clearly see the direction of the theory’s development.

Example 6.9. (Examples vs. general theory: Coordinate systems)

A system of unit vectors \( \hat{\rho}, \hat{\phi}, \hat{z} \) [Symon [99, p.95, l.−1; Fig. 3.22]] gives the orthonormal basis of cylindrical polar coordinates; a system of unit vectors \( \hat{r}, \hat{\theta}, \hat{\phi} \) [Wangness [106, p.31, l.−11−l.−10; p.32, Figure 1-39]] gives the orthonormal basis of spherical coordinates. For the general case, see Arfken & Weber [2, p.8, Exercise 2.1.1].

Example 6.10. (Using the standard form of conicoids to simplify the proofs of theorems about symmetric matrices)

Consider the system of equations given in Bell [6, p.216, (1), (2), & (3)]. Let

\[ r = \text{rank of the coefficient matrix } \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \]

and

\[ r' = \text{rank of the argumented matrix } \begin{pmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \end{pmatrix} \]
\[ r' = \text{rank of the argumented matrix } \begin{pmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \end{pmatrix}. \]

Let \[ D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad \text{and} \quad \Delta = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}. \]

Bell [6, p.220, l.21–l.22] says that \([(r = 2, r' = 3) \Rightarrow (\text{the conicoid is a paraboloid})]\). Fine–Thompson [33, p.283, Table] says that \([(D = 0, \Delta \neq 0) \Rightarrow (\text{the conicoid is a paraboloid})]\). Hence \((r = 2, r' = 3) \Leftrightarrow (D = 0, \Delta \neq 0)\).

**Proof.** Since the rank and the determinant of a matrix are invariant under nonsingular linear transformations, we may use standard forms of conicoids to prove this theorem. Using a complete list [Fine–Thompson [33, §237]] of standard forms of conicoids to check if there is any form satisfies the property \((r = 2, r' = 3)\), we find that only the standard form \(\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}\) of a paraboloid satisfies this property. Consequently, paraboloids can be characterized by the property \((r = 2, r' = 3)\). \(\Box\)

**Example 6.11.** (Trinity of consecutive points, approach to the same point, and contact of higher order)

The definition given in Bell [6, p.279, l.12–l.10] is a rigorous statement of the definition given in Weatherburn [110, vol.1, p.12, l.6–l.8]. The definitions of osculating plane, osculating circle, osculating sphere given in Bell [6, p.279, l.12–l.10; p.292, l.1–l.3; l.1–l.1] are based on the same idea of osculation. The second approach is heuristic, systematic and unified. In contrast, the definitions given in Kreyszig [61, Table 10.1; p.51, l.4 & l.17] look artificial. The common idea of these three definitions becomes vague in the third approach. The important step for the construction process (the circle \(PQR\) [Bell [6, p.292, l.2]] or the sphere \(PQRS\) [Bell [6, p.292, l.2]]) in the second approach is lost in the third approach and cannot be restored by using the results in the third approach alone. From hindsight, consecutive points can be considered a brief expression for contact of higher order [Kreyszig [61, p.50, l.6–p.51, l.6]]. The contact of second or third order can easily be generalized to the \(n\)th order.

**Example 6.12.** (Regarding a rigorous proof as an improvement)

Sneddon [92, p.20, l.1–p.21, l.7] provides a rigorous proof of Bell [6, p.318, l.9–l.1].

**Remark.** When reading classical books, one should be familiar with their frequent mistakes and should know how to correct them. Otherwise, one cannot appreciate these books. In most cases, the statement of a theorem is correct, but the author fails to provide a rigorous proof. If we read Bell [6, p.318, l.9–l.1] alone, we do not know the proof can be improved. Likewise, if we read Sneddon [92, p.20, l.1–p.21, l.7] alone, we do not know it is an improvement of a classical theorem.

**Example 6.13.** (Only after a construction is tailored to our needs can it solve the problem effectively)

In order to prove the countable additivity of \(P\), the construction given in the proof of Borovkov [14, p.31, Theorem 1] is tailored to our needs and effectively meets our goal. The construction is lean, simple and clear. Note that we can take \(\tilde{B}_n\) such that \(B_{n+1} \subset \tilde{B}_n \subset B_n\). In contrast, the step IV of the proof of Rudin [88, p.42, Theorem 2.14] provides a second construction, but this construction is too general and abstract to be practical. The construction is burdened by unnecessary equipment: the existences in the proofs of Rudin [88, p.37, Theorem 2.5; p.38, Theorem 2.7] are provided by topological axioms; the construction in the proof of Rudin [88, p.40, Lemma p.2.12] is too complicated to be useful in practice. Consequently, it is difficult to apply the second construction to practical cases. A big apparatus may be impressive from the theoretical viewpoint. However, it not only fails to point out the key idea, but also is useless in applications.
Example 6.14. (Only after a proof is tailored to our needs may we grasp the key idea)

The more direct a proof is, the more powerful it is. The less deviated or involved an argument is, the clearer the key idea becomes. There are three proofs of Borovkov [14] p.435, Lemma 3]. Borovkov [14] p.436, l.1–l.10] provides the first proof. The second one follows from Borovkov [14] p.115, Lemma 2]. The third one follows from Rudin [88] p.176, Theorem 8.17; Theorem 8.16; p.132, Theorem 6.11]. The first proof is the most direct of the three.

Example 6.15. (An index set must be chosen properly: one more candidate would be too many and one less would be too few)

The ch.f. of a random variable uniquely determines its distribution function [Borovkov [14] p.139, 1.18–1.19]].

Proof. Let \( n = 2 \) and \( \Delta = (a_1, b_1) \times (a_2, b_2) \) [Borovkov [14] p.130, 1.6–1.7]]. In order to define \( F_\xi(0) \), we need to find \( \Delta_1 \) so that \( \cup_1 \Delta_i = (-\infty, 0) \times (-\infty, 0) \), where \( \Delta_i \)’s are mutually disjoint. This will ensure that except for countable \( \iota \), \( P_\xi(\partial \Delta_1) = 0 \). Then, by left-continuity of \( F_\xi \) and the inversion formula [Borovkov [14] p.130, 1.6–1.9]], we may define \( F_\xi(0) = \sup_1 P(\xi \in \Delta_i) \).

Case I. (Improper choice: one more candidate would be too many)

If we define \( F_\xi(0) = \sup_{\Delta \subseteq (-\infty, 0) \times (-\infty, 0)} P(\xi \in \Delta) \), at best we define only a unsolved problem. This is because many recruited candidates are unqualified, but we put them into consideration and have no way to rid them in order to satisfy the condition. Confucius says, “Only by careful distinguishing what one knows from what one dones’t may one have a deeper understanding.”

Case II. (Wrong choice: one less would be too few)

\( \forall k \in \mathbb{N} \), let \( \Delta_k = (-k, -\frac{1}{k}) \times (-k, -\frac{1}{k}) \). Suppose we define \( F_\xi(0) = \sup_1 P(\xi \in \Delta_k) \). Since \( P_\xi(\partial \Delta_k) \) may not be 0, there may be not enough \( \Delta_k \)’s that can satisfy the condition \( \cup P_\xi(\partial \Delta_k) = 0 \Delta_k = (-\infty, 0) \times (-\infty, 0) \). Thus, we choose too few candidates.

Case III. (Proper choice)

\( \forall x \geq 1 \), let \( \Delta_x = (-x, -\frac{1}{x}) \times (-x, -\frac{1}{x}) \). By Rudin [88] p.17, Theorem 1.19(d)], we define \( F_\xi(0) = \sup_{\Delta \subseteq \mathbb{N}} P(\xi \in \Delta) \). Then \( \Delta_x \)’s are mutually disjoint. Consequently, except for countable \( x \), \( P_\xi(\partial \Delta_x) = 0 \).

Remark. Mathematics discusses the process of finding a solution rather than just proves that a given solution is true.

Example 6.16. (The strong law of large numbers and the central limit theorem [Borovkov [14] p.151, Theorem 1; p.152, Theorem 2]; Lindgren [68] p.155, Khintchine’s theorem; p.158, central limit theorem])

1. The proofs of both Lindgren [68] p.155, Khintchine’s theorem; p.158, central limit theorem] and Borovkov [14] p.151, Theorem 1; p.152, Theorem 2] are essentially the same except that the former proofs use the lemma given in Lindgren [68] p.156, 1.6–1.8] while the latter proofs do not. It is easy for the former proofs to be generalized to the multidimensional case, but it is difficult for the latter proofs. Furthermore, \(-\frac{\epsilon}{T} + o(1) \rightarrow -\frac{\epsilon}{T} \) given in Borovkov [14] p.153, 1.3] is incorrect because \( o(1) \) refers to \( t \rightarrow 0 \) rather than \( n \rightarrow \infty \).

2. (Stronger convergences) The proofs of both Lindgren [68] p.155, Khintchine’s theorem; p.158, central limit theorem] and Borovkov [14] p.151, Theorem 1; p.152, Theorem 2] use Borovkov [14] p.132, Theorem 2]. Therefore, the coverages in both Khintchin’s theorem and the central limit theorem are essentially weak convergences. The proof of Borovkov [14] p.151, Theorem 1] gives the weak convergence \( F_{S_n/n} \Rightarrow a \). By Lindgren [68] p.154, Theorem B], the weak convergence can be strengthened to
the convergence in probability \( S_n/n \xrightarrow{P} a \). The strong law of large numbers [Chung 14, p.133, Theorem 5.4.2 (8)] strengthens the convergence in probability \( S_n/n \xrightarrow{P} a \) further to the almost sure convergence \( S_n/n \xrightarrow{a.s.} a \) [Borovkov 14, p.151, 1.–11l]. The proof of Borovkov 14, p.152, Theorem 2] gives the weak convergence \( F_{\zeta_n} \Rightarrow \Phi \). By Borovkov 14, p.116, Theorem 6], we have the pointwise convergence \( F_{\zeta_n}(x) \to \Phi(x) (x \in \mathbb{R}) \). By Parzen 75, Exercise 5.2], \( F_{\zeta_n}(x) \to \Phi(x) \) uniformly in \( x \in \mathbb{R} \).

Remark. The strong law of large numbers for Bernoulli scheme follows from Borovkov 14, p.91, Theorem 2; p.109, Theorem 1].

(3) If the metric space \( S \) given in Billingsley 9, p.3, 1.9] is the real line \( \mathbb{R} \), then the proof given in Lindgren 68, p.154, 1.–5–p.155, 1.5] is more intuitive than that given in Billingsley 9, p.27, 1.5–1.15].

Remark. For the right side of the inequality given in Lindgren 68, p.155, 1.2], note that \( P(Y_n = k – \varepsilon) = 0 \) [Billingsley 9, p.26, Theorem 2.1(iii)]].

(4) The motivation to choose \( Z_n \) in formulating the central limit theorem [Lindgren 68, p.157, 1.8–l.3].

I. Choose \( Z_n \) instead of \( S_n \) to keep track of the shape of the limiting distribution function.

II. Choose \( Z_n \) instead of \( Y_n \) to avoid the singularity of the limiting distribution function.

By the weak law of large numbers, \( F_{\zeta_n} \Rightarrow F_{\zeta_n^X} \), where \( F_{\zeta_n^X} \) has a single jump at \( EX \).

III. Standization (mean = 0, var = 1) that keeps the limiting distribution function from shrinking or expanding leads us from \( Y_n \) to \( Z_n \) naturally.

(5) (Motivation for using characteristic functions; key points vs. details; natural proofs; physical meanings)

Both Reif 84, p.35, 1.6–p.40, 1.8] and Borovkov 14, p.75, Theorem 7; §5.1–5.3; §5.5; §7.1–§7.4; §7.6; §8.1–§8.2] discuss the strong law of large numbers and the central limit theorem. The former indicates the motivation for using characteristic functions to prove these theorems [Reif 84, p.36, 1.1–1.17]] and reveals that the key idea of proving these theorems is simple, original and excellent [Reif 84, p.36, 1.17–l.1–11]. However, the former lacks details; its statements are crude; its proofs are not rigorous. Although the latter provides details, accuracy, and rigor, its proofs lack motivations and its key points are vague. The way to keep the merits of both approaches is to select the key statements in the former and find their corresponding rigorous ones in the latter.

I. \( P(x)dx \) [Reif 84, p.35, 1.10]] \to dF(x) [Borovkov 14, p.29, 1.–17]. For the latter expression, every component (the set of elementary outcomes, \( \sigma \)-algebra, probability) of the probability space [Borovkov 14, p.17, Definition 6]] is clearly specified. Thus, we have a rigorous mathematical structure ready to hand.

II. Reif 84, p.35, (1-10-2)] \to Borovkov 14, p.54, 1.7–l.16; p.126, 1.11]. This can lead to the equality given in Borovkov 14, p.97, 1.8].

III. By using Dirac \( \delta \) function, we obtain Reif 84, p.35, (1-10-2) & (1-10-3) = Reif 84, p.36, (1-10-4)]. Reif 84, p.36, (1-10-5)] \to Borovkov 14, p.126, 1.4–l.6; p.130, (5)]. Reif, 84, p.36, (1-10-5)] which leads to Reif 84, p.36, (1-10-6)] gives the reason why we should use characteristic functions to prove the strong law of large numbers and the central limit theorem. Consequently, do not consider the inversion formula a unnatural thing. In fact, based on the physical consideration given in Reif 84, p.36, 1.1–1.17], only through the use of characteristic functions and the inversion formula may we have a simple, natural and general [Reif 84, p.37, 1.–7–l.1–6] method of dealing with the convergence of the sum of a sequence of independent identically distributed random variables. The approaches given in Borovkov 14, p.97, 1.12–p.99, 1.11] and in the proof of Chung 21, p.114, Theorem 5.2.2] are artificial, while the proofs of Borovkov 14, p.151, Theorem 1; p.152, Theorem 2] are natural.

IV. For the Riemann–Lebesgue lemma [Borovkov 14, p.129, 8]; Rudin 88, p.197, Theorem 9.6]]], Reif 84, p.38, 1–l.4] provides its physical meaning and the motivation for its formulation. The proof given
in Reif [84, p.38, Remark] is not as good as the proof given in https://en.wikipedia.org/wiki/Riemann%E2%80%93Lebesgue_lemma.

V. Reif [84, p.38, l.1–l.12] provides the motivation for formulating the central limit theorem [Reif [84, p.39, l.–7]; Borovkov [14 p.152, Theorem 2]].

**Example 6.17.** (Indigo blue is extracted from the indigo plant but is bluer than the plant it comes from)

Most mathematical theorems do not come from nowhere. A new theorem is often a supplement, a stronger version, an analog or an extension of an old theorem. The generation of this kind of derivatives makes up a significant part of the development of a theory. Thus, indigo blue is extracted from the indigo plant but is bluer than the plant it comes from. The following evidences convince us that we should learn to control rather than follow the flow of a proof.

I. Supplements: Chung [21 p.133, Theorem 5.4.2, (9)] is a supplement of Chung [21 p.133, Theorem 5.4.2, (8)]. The former discusses Case $\mathbb{E}(X_1) = \infty$, while the latter discusses Case $\mathbb{E}(X_1) < \infty$.

II. Stronger versions: Chung [21 p.133, Theorem 5.4.2] is stronger than Chung [21 p.114, Theorem 5.2.2]. The convergence in probability of the latter is strengthened to the almost sure convergence of the former.

III. Extensions: Chung [21 p.134, Theorem 5.4.3] is an extension of Chung [21 p.133, Theorem 5.4.2, (9)] because the hypothesis of the former is more flexible than that of the latter. Similarly, Chung [21 p.121, Theorem 5.3.1] is an extension of Chung [21 p.50, Chebyshev’s inequality] and Chung [21 p.116, Theorem 5.2.3] is an extension of Chung [21 p.114, Theorem 5.2.2]. The example of strengthening given in II is a special case of Chung [21 p.126, Theorem 5.3.4]. Since Borovkov [14 p.151, Theorem 1] is the same as Chung [21 p.114, Theorem 5.2.2], we expect the proof of Borovkov [14 p.151, Theorem 1] and that of Chung [21 p.133, Theorem 5.4.2, (8)] should be similar, but they are actually different. In order to organize the structure of the proof of Chung [21 p.133, Theorem 5.4.2, (8)], we may use Borovkov [14 p.151, Theorem 1] and Chung [21 p.126, Theorem 5.3.4] to prove Chung [21 p.133, Theorem 5.4.2, (8)]. The proof provided by this method is more compatible with that of Borovkov [14 p.151, Theorem 1] than that of Chung [21 p.133, Theorem 5.4.2, (8)].


Remark 1. The proof of Chung [21 p.133, Theorem 5.4.2, (8)] and that of Loève [69, p.251, l.15–p.252, l.6] are essentially the same.

Remark 2. (Control the work flow by dividing it into several applicable sections) It seems magical that Chung [21 p.121, Theorem 5.3.1] makes the hypothesis of Borovkov [14 p.75, (13)] more flexible [Chung [21 p.121, l.–3–l.–1]] . If we compare Chung [21 p.122, l.–2–l.–1] with Borovkov [14 p.75, l.4], we find that their key ideas are essentially the same except that the former divides the work flow into several applicable sections [Chung [21 p.122, l.10]]. This technique of segmentation designed for independence is also used in the proofs of Chung [21 p.123, Theorem 5.3.2; p.126, Theorem 5.3.4].

**Example 6.18.** (An advanteous viewpoint can facilitate the calculations in a proof)

The proof [Pontryagin [81 p.50, l.–22–p.52, l.–4]] of the first part of Pontryagin [81 p.52, (B)] originates from the viewpoint of differentiation [Pontryagin [81 p.50, l.–14]]. In contrast, the proof [Hartman [48 p.324, l.1–l.16]] of Hartman [48 p.324, (1.15)(i)] originates from the viewpoint of changing variables [Hartman [48 p.324, l.11]]. The latter viewpoint can facilitate the calculations in the proof of Pontryagin [81 pp.50–51, Theorem 5].

**Example 6.19.** (Method vs. calculation for solutions)

$u(t)$ and $v(t)$ are linearly independent solutions of (2.1) if and only if $c \neq 0$ in (2.7) [Hartman [48 p.327, l.8–l.9]]
Proof with theory as a guide. \( c \neq 0 \)
\[ \Leftrightarrow \det X(t) \neq 0 {\text{[Hartman [48 p.326, (2.7)]}}} \]
\[ \Leftrightarrow (u(t), p(t)u'(t)) \text{ and } (v(t), p(t)v'(t)) \text{ are linearly independent} \]
\[ \Leftrightarrow (u(t) \text{ and } v(t) \text{ are linearly independent}) \text{[Hartman [48 p.326, (iv)]].} \]


Remark. The first proof helps us grasp the key points, while the second proof helps us understand the original approach. The second proof requires patience and tricks \[\text{[Hartman [48 p.326, l.12–l.13]]} \] to deal with nuisances \[\text{[Ince [54, p.117, l.14; signs of cofactors]}} \] and difficult points \[\text{[Ince [54, p.117, l.–20; continuity].}} \]

Example 6.20. (Boltzmann’s entropy formula)

Most textbooks in statistical mechanics define entropy artificially as \( S = k \ln \Omega \). Actually, it is more heuristic to ask how to derive this formula. Consider the isothermal expansion of ideal gas.

\[ \Delta E = 0 \text{[Reif [84, p.126, (3-12-11)]]} \]
\[ \Rightarrow \Delta Q = \Delta W = \int V_i \, \rho \, dV \]
\[ = NkT \ln \frac{V_f}{V_i} \text{[Reif [84 p.125, (3-12-8)]]}. \]

\[ dS = \frac{\Delta Q}{T} = k \ln \frac{Q(E,V_f)}{Q(E,V_i)} \text{[Reif [84, p.64, (2-5-14)]].} \]

It is more natural to relate entropy in thermodynamics to microstates at this moment and in this way.

Example 6.21. (An easy way to make the discussion of \( \delta(x) \) rigorous)

\[ \delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \text{ on } [-\pi, \pi] \text{[Jackson [55 p.125, 1.13, (3.139)(ii)]].} \]

1st proof. \( D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)} \text{[Rudin [86 p.174, (77)]}] \)
\[ \Rightarrow D_n(2x) = \frac{\sin((2n+1)x)}{\sin(x)} \]
\[ \Rightarrow \lim_{n \to \infty} D_n(2x) = \delta(x) \text{[Born–Wolf [13 p.897, (20) & (21)]]} \]
\[ \Rightarrow \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{in(2x)} = \delta(x) \text{[Rudin [86 p.174, (76)(i)]]} \]
\[ \Rightarrow \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{inx} = \delta(x/2) = 2\delta(x) \text{[Cohen–Tannoudji–Diu–Laloë [23 vol. 2, p.1471, (20)].]} \]

2nd proof. Let \( f \) be a continuous function of bounded variation with a period \( 2\pi \).
\[ s_n(f;x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_n(t) \, dt \text{[Rudin [86 p.175, (82)]]} \]
\[ \Rightarrow \int_{-\pi}^{\pi} f(x-t) \left( \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{int} \right) \, dt = \lim_{n \to \infty} s_n(f;x) = f \text{ [Royeren [85 p.232, Proposition 18]; Zygmund [113 vol. 1, p.57, Theorem (8.1)(i)]]} \]
\[ = \int_{-\pi}^{\pi} f(x-t) \sum_{q=-\infty}^{\infty} \delta(t-2\pi q) dt \text{[Cohen–Tannoudji–Diu–Laloë [23 vol. 2, p.1473, (31)]]} \]
\[ = \int_{-\pi}^{\pi} f(x-t) \delta(t) dt \]
\[ = \int_{-\pi}^{\pi} f(x-t) \delta(t) dt \text{[} \delta(t) = 0 \text{ if } t \notin [\pi, \pi]. \text{]} \]
\[ \Rightarrow \delta(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{imt} \text{ a.e. on } [-\pi, \pi] \text{ [Rudin [88 p.31, Theorem 1.39(b)]].} \]

Remark 1. Strictly speaking, the first proof is not good because it fails to consider the requirement given in Reif [84 p.614, l.–7]. Both proofs are not rigorous because a generalized function should not be treated as a function. In order to correct the problem, we should use the language of functional analysis. Actually, the required supplement is not much. For the discussion of the Dirac delta function, it requires only Rudin [87 p.142, l.–6, (1); p.155, (2) & (5)] to bridge the gap between a function and a generalized function.  

59
For the discussion of derivatives of $\delta$ [Cohen-Tannoudji–Diu–Laloë [23] vol. 2, p.1476, b]], it requires only Rudin [87] p.133, (1), (2) & (3) to bridge the gap between a function and a generalized function.

**Remark 2.** The discussion given in Cohen-Tannoudji–Diu–Laloë [23] vol. 2, Appendix II is not good because a generalized function should not be treated as a function. It requires a rigorous theory to correct and support the discussion. The theory contained in Rudin [87, chap. 6] is rigorous, but it fails to directly apply to the Dirac delta generalized function. Many physicists fail to understand the theory. This is the reason why theory and applications are easily disconnected. Therefore, it is important to identify their connections.

**Example 6.22.** (A good theorem should provide complete information)

(A) If we treat $\mathbb{R}^2$ as a topological subspace of its one-point compactification $S^2$ and denote the boundary relative to $S^2$ as $\partial_{\infty}$, then the geometric meaning of “$y(t)$ tends to $\partial E$ as $t \rightarrow \omega^+$” given in Hartman [48] p.13, 1.5-1.6] is “$(t,y(t))$ tends to $\partial_\infty E$ as $t \rightarrow \omega^+$”. A good theorem should provide complete information. In the above sense, the conclusion of Hartman [48] pp.12–13, Theorem 3.1] gives a complete geometric picture, while the result given in Hirsch–Smale–Devaney [27] p.398, 1.10–1.9] to which Hirsch–Smale–Devaney [27] p.398, Theorem] leads fails to completely describe what it should.

(B)

**Lemma.** Let $f(t,y)$ be continuous on a $(t,y)$-set $E$. Let $y = y(t)$ be a solution of $y' = f(t,y)$ on $[a, \delta)$, $\delta < \infty$, for which $\exists y_n \in [a, \delta): (\lim_{n \to \infty} t_n = \delta$ and $\lim_{n \to \infty} y(t_n) = y_0)$. If $f(t,y)$ is bounded on the intersection of $E$ and a vicinity of the point $(\delta, y_0)$, then $\lim_{t \to \delta} y(t) = y_0$ [Hartman [48] p.13, Lemma 3.1].

**Proof.** I. By hypothesis, we may take a small $\varepsilon > 0$, and a large $M_\varepsilon > 0$ such that $|f(t,y)| \leq M_\varepsilon$ for $(t,y) \in E \cap \{(t,y) | 0 \leq \delta - t \leq \varepsilon, |y - y_0| \leq \varepsilon\}$.

II. Take a large $n$ such that $0 < \delta - t_n \leq \frac{\varepsilon}{2M_\varepsilon}$ and $|y(t_n) - y_0| \leq \varepsilon/2$. Then

III. $\forall t_0 \leq t < \delta$. $|y(t) - y(t_n)| < M_\varepsilon (\delta - t_n)$.

**Proof.** Assume that III were false. Then $\exists t^* \leq t < \delta$. $|y(t) - y(t_n)| \geq M_\varepsilon (\delta - t_n)$. Let $t^1 = \min \{t \in [a, \delta) : |y(t) - y(t_n)| = M_\varepsilon (\delta - t_n)\}$. Then

1. $t_0 < t^1 < \delta$.

2. $|y(t^1) - y(t_n)| = M_\varepsilon (\delta - t_n) \leq \varepsilon/2$ (by II).

3. $\forall t_0 \leq t < \delta, |y(t) - y_0| \leq |y(t) - y(t_n)| + |y(t_n) - y_0| < M_\varepsilon (\delta - t_n) + |y(t_n) - y_0|$ (by the definition of $t^1$)

4. $\forall t_0 \leq t < \delta, |y(t)| = |f(t,y(t))| \leq M_\varepsilon$. 

60
Proof.

\[ \delta - t \leq \delta - t_n \leq \frac{\varepsilon}{2M} \quad \text{(by II)} \]
\[ \leq \varepsilon. \]

By 3, \(|y(t) - y_0| \leq \varepsilon\). The result follows from I.

5.

\[ |y(t^1) - y(t_n)| \leq M_\varepsilon(t^1 - t_n) \quad \text{(by 4)} \]
\[ < M_\varepsilon(\delta - t_n) \quad \text{(by 1)}. \]

This would contradict the definition of \(t^1\).

Remark. The proof of Hartman [48, p.13, Lemma 3.1] is hard to read because all it contains is a series of formulas with little documentation.

(C) \(y(t)\) tends to the boundary \(\partial E\) of \(E\) as \(t \to \omega^+\) [Hartman [48, p.13, 1.2–1.3]].

Proof. I. Because \((b_k, y(b_k)) \notin \bar{E}_{n(k)}, (b_1, y(b_1)), (b_2, y(b_2)), \ldots\) is either unbounded or has a cluster point on the boundary \(\partial E\) of \(E\) [Hartman [48, p.13, 1.12–1.11]].

II. Assume the statement “\(y(t)\) tends to the boundary \(\partial E\) of \(E\) as \(t \to \omega^+\)” were false. Then \(\exists t_n \in [a, \omega^+) : \lim_{t \to \omega^+} (t_n, y(t_n)) = (\omega^+, y_0) \in \bar{E}_m\).

1. Consequently, \(f\) is bounded on the intersection of \(E\) and a vicinity of the point \((\omega^+, y_0)\). That is, \(\exists c \in [a, \omega^+), M > 0 : (c \leq t < \omega^+) \Rightarrow |y'(t)| = |f(t, y(t))| \leq M\). Thus, \(y(t)\) is uniformly continuous on \([c, \omega^+)\). We may define \(y(\omega^+) = \lim_{t \to \omega^+} y(t)\). By Dugundji [28, p.302, Theorem 5.2], the extension of \(y(t)\) is uniformly continuous on \([a, \omega^+)\).

2. \(y(t) : [a, \omega^+] \to \mathbb{R}\) is differentiable at \(\omega^+\) and is a solution of \(y'(t) = f(t, y(t))\) on \([a, \omega^+]\).

Proof.

\[ y(t) = y(a) + \lim_{t \to \omega^+} \int_a^t y'(s) ds \quad \text{(by definition)} \]
\[ = y(a) + \lim_{t \to \omega^+} \int_a^t f(s, y(s)) ds \]
\[ = y(a) + \int_a^{\omega^+} f(s, y(s)) ds \quad \text{(Rudin [88, p.27, Theorem 1.34])}. \]

Consequently, \(\forall t \in [a, \omega^+], y(t) = y(a) + \int_a^t f(s, y(s)) ds\) and \(y'(t) = f(t, y(t))\).

3. Since \(\text{dist}(\bar{E}_m, \partial E) \geq 1/m\), by Hartman [48, p.11, Corollary 2.1], there exists a \(\delta > \omega^+\) such that the solution \(y(t)\) on \([a, \omega^+]\) can be extended to \([a, \delta]\). This would contradict the fact that \([a, \omega^+]\) is the right maximal interval.
Example 6.23. (The general method of finding a Green function’s eigenfunction expansion: using symmetry)

\[ G(x, x') = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin \left( \frac{l\pi x}{a} \right) \sin \left( \frac{l\pi y}{a} \right) \sin \left( \frac{m\pi z}{b} \right) \sin \left( \frac{m\pi z'}{b} \right) \frac{\sinh\left(\frac{l \pi x'}{a}\right) \sinh\left(\frac{m \pi z'}{b}\right)}{K_{lm} \sinh(K_{lm} z')}, \]

where \( K_{lm} = \pi \left( \frac{l^2}{a^2} + \frac{m^2}{b^2} \right)^{1/2} \) [Jackson [55] p.129, (3.168)].

Proof. I. Let \( G(x, x') = \frac{16\pi}{ab} \sum_{l,m=1}^{\infty} \sin \left( \frac{l\pi x}{a} \right) \sin \left( \frac{l\pi y}{a} \right) \sin \left( \frac{m\pi z}{b} \right) \sin \left( \frac{m\pi z'}{b} \right) g(l, m, z, z') \) (by symmetry and a theorem similar to Coddington–Levinson [22, p.197, Theorem 4.1]).

\[ \nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} = -4\pi \delta(x - x') \delta(y - y') \delta(z - z') \]

[Cohen-Tannoudji–Diu–Laloe [23], vol. 2, p.1477, (59)]

\[ = -4\pi \delta(z - z') \sum_{l,m=1}^{\infty} \frac{4}{ab} \sin \left( \frac{l\pi x}{a} \right) \sin \left( \frac{l\pi y}{a} \right) \sin \left( \frac{m\pi z}{b} \right) \sin \left( \frac{m\pi z'}{b} \right) \]


Because \( \nabla^2 G = -4\pi \delta(x - x') \) [Jackson [55] p.120, (3.116)] and \( \{ \sin \left( \frac{l\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right) \}\}_{lm} \) are linearly independent,

\[ \frac{\partial^2 G}{\partial z^2} - K_{lm} G = -\delta(z - z'). \]

II. The desired result follows from Birkhoff–Rota [10] p.286, (67)].

Remark. (The general method of finding a Green function’s eigenfunction expansion: using symmetry)

In order to reduce the problem of finding a 3-dim Green function to the problem of finding 1-dim Green
function, we should summarize the proof of Jackson [55] p.121, 1.2, (3.120), the proof of Jackson [55] p.125, (3.141), and Part I of the above proof as follows: Put the unit charge into the volume of interest. Let \( x' \) be its position. Let \( x, y, z \) be the Green function’s three variables. Now use \( z' \) to divide the volume into two regions:

I. \( \{ x | z < z' \} \); II. \( \{ x | z > z' \} \). In these two regions, the Poisson equation \( \nabla^2 G = -4\pi \delta(x - x') \) is reduced to the Laplace equation \( \nabla^2 G = 0 \). Let \( \{ \phi_{lm}(x, y) \}_{lm} \) be the basis of the solution space. By symmetry and a theorem similar to Coddington–Levinson [22] p.197, Theorem 4.1], we have

\[ G(x, x') = \sum_{lm} g_{lm}(z, z') \phi_{lm}(x, y) \phi_{lm}(x', y'). \]

By substituting this expression for \( G \) into \( \nabla^2 G = -4\pi \delta(x - x') \) and using Cohen-Tannoudji–Diu–Laloe [23] vol. 1, p.100, (A-32)], we will obtain the equation for 1-dim Green function.

In the last paragraph, we have used the fact that \( G \) is symmetric in \( (x, y) \) and \( (x', y') \). In order to find the solutions of the equation for the 1-dim Green function, we should use the fact that \( G \) is symmetric in \( z \) and \( z' \). This usage of symmetry is more subtle, more refine, and more interesting than the previous one. In view of the example given in Coddington–Levinson [22] p.222, 1.9–1.14], the algebraic methods of sloving the boundary value problems such as Jackson [55] p.121, 1.2, (3.120)] (\( g_l(0) \) is finite; \( g_l(\infty) = 0 \), Jackson [55] p.125, (3.141)] (\( g_n(0) \) is finite; \( g_n(\infty) = 0 \), and Birkhoff–Rota [10] p.286, Theorem 12) are essentially the same. No wonder Jackson [55] p.120, 1.1–1.9] and Jackson [55] p.125, 1.11–1.3] have similar geometric interpretation for for region I: \( \{ x | z < z' \} \) and region II: \( \{ x | z > z' \} \). Jackson should have quoted Birkhoff–Rota [10] p.286, Theorem 12] whenever necessary instead of repeating its proof many times.

Example 6.24. (The Ritz method is an effective tool for studying Sturm–Liouville Problems [Fomin–Gelfand [35] pp.198–205, §41])

I. Calculus tools for finding extrema of functions: Kaplan [58] §2.19; §2.20.

Tools in calculus of variations for finding extrema of functionals: Direct methods (the Rayleigh–Ritz method; the method of finite differences) and using Euler equations [Courant–Hilbert [26] vol.1, chap. IV, §2].

method [Fomin–Gelfand [35] p.196, Theorem]: construct a complete sequence of functions $\varphi_n$ as in Fomin–Gelfand [35] p.195, (8)); this sequence allows us to reduce the problem of finding the minimum of the functional $J_1[y]$ to the problem of finding the minimum of the function $J(\alpha_1 \varphi_1 + \cdots + \alpha_n \varphi_n)$ of the $n$ variables $\alpha_1, \cdots, \alpha_n$ [Fomin–Gelfand [35] p.195, (10)]]. Thus, it suffices to calculate $y_n$ given in Fomin–Gelfand [35] p.196, 1.13–1.14] by using calculus tools for finding extrema for functions.

III. The existence of $\lambda^{(1)}_0$ given in Fomin–Gelfand [35] p.200, (24)] is more constructive and effective than the existence of $\mu_0$ given in Coddington–Levinson [22] p.195, 1.−9].

**Explanation.** (A).

1. $M$ defined as in Fomin–Gelfand [35] p.199, 1.5] can be computed by calculus.

2. For a system’s solution, we may replace its function (uncountable) form $y(x)$ with its sequence (countable) form $\alpha_k$ as in Fomin–Gelfand [35] p.199, (18)]. Thus, $J[y]$ is transformed to $J(\alpha_1 \varphi_1 + \cdots + \alpha_n \varphi_n)$, a quadratic form in $\alpha_1, \cdots, \alpha_n$. The minimum of the latter can be computed by the methods given in Kaplan [58] §2.19; §2.20].

3. Define $\lambda^{(1)}_n, y_n$ $(n = 1, 2, \cdots)$ as in [Fomin–Gelfand [35] p.199, l.−10–l.−7)]. Then $\lambda^{(1)}_{n+1} \leq \lambda^{(1)}_n$ [Fomin–Gelfand [35] p.200, (23)]. Define $\lambda^{(1)}_0$ as in Fomin–Gelfand [35] p.200, (24)]. After obtaining $\lambda^{(1)}_1, \cdots, \lambda^{(1)}_n$, we know $\lambda^{(1)}_n$ is between $\lambda^{(1)}_{n+1}$ and the lower bound of $\{\lambda^{(1)}_0\}$. Thus, the possible range of $\lambda^{(1)}_0$ is getting shorter and shorter as the process goes on. In Fomin–Gelfand [35] p.201, 1.−14–p.203, 1.−3], we use the method of Lagrange multipliers to obtain Fomin–Gelfand [35] p.203, (36)] and then use Fomin–Gelfand [35] p.201, Lemma 2] to prove Fomin–Gelfand [35] p.202, (32)].

(B). In contrast, $\mu_0 = \sup_{\|u\|=1} \{|(\mathcal{F}u, u)| \quad (u \in C\text{ on } [a, b]\}$ [Coddington–Levinson [22] p.195, 1.2; 1.−9]]. The existence of supremum is derived from reduction to absurdity [Rudin [86] p.11, 1.−17–l.−16]. We have no way to know its location on the real line. Furthermore, as we collect more elements of the index set $(u \in I)$ and find $\sup \{(\mathcal{F}u, u)| u \in I\}$, this procedure will not help narrow down the search scope of the final supremum.

**Remark.** Based on (A), one can easily create a effective computer program to find $\lambda^{(1)}_0$. However, the idea given in (B) is useless for one to find $\mu_0$ using a computer. Mathematicians should put more effective stuff than the content given in Coddington–Levinson [22] p.194, l.−6–p.197, 1.8] into mathematical textbooks.

IV. By III, $\lambda^{(1)}, \lambda^{(2)}, \cdots; y^{(1)}, y^{(2)}, \cdots$ [Fomin–Gelfand [35] §41.4)] can be effectively calculated using the method of Lagrange multipliers, while the existence of $\mu_k$ $(k = 0, 1, 2, \cdots)$ given in Coddington–Levinson [22] p.195, 1.−9–p.196, 1.−2] is derived from the $(k+1)$th level of reduction to absurdity. Furthermore, that the process of finding $\mu_0, \mu_1, \cdots$ can be continued is proved by reduction to absurdity [Coddington–Levinson [22] p.197, 1.1–l.−7], while that the process of constructing $\lambda^{(1)}, \lambda^{(2)}, \cdots$ can be continued because each step of the process satisfies the conditions of the method of Lagrange multipliers.

**Example 6.25.** (Derivation of the equation of the vibrating membrane)

In order to effectively solve a problem, we must quickly understand the context within the minimum effort, and then directly attack the heart of the matter. The local consideration given in [§6.1; http://personal.egr.uri.edu/sadd/mce565/Ch6.pdf] provides a simple derivation of the equation of the vibrating membrane. Newton’s law is the only requirement. Considering a circular membrane with polar coordinates only complicates the circumstance [§43.3; https://theses.lib.vt.edu/theses/available/etd-08022005-145837/unrestricted/Chapter4ThinPlates.pdf].

Fomin–Gelfand [35] p.164, (48)] is derived from the viewpoint of the calculus of variations. The derivation starts with the Hamiltonian principle and ends with the Euler equation. The principle acts like an axiom and the equation acts like a theorem. The formal development makes it difficult to see the key point. The
benefit of this approach is to provide the boundary condition [Fomin–Gelfand [35, p.164, (51)] simultaneously.

\[
\int \int \mathcal{R} \left[ \frac{\partial}{\partial x}(u_x \psi) + \frac{\partial}{\partial y}(u_y \psi) \right] dxdy = \int \Gamma \frac{\partial u}{\partial n} \psi ds [\text{Fomin–Gelfand [35, p.163, l.}12–l.10].
\]

The global consideration given in [§Vibrating Membranes; http://www.math.iit.edu/~fass/Notes461_Ch7Print.pdf] increases the difficulties of the following problems:

1. Finding the tensile force \( F_T \) [p.6, l.4].
2. The balance of forces [p.7, (1)].
3. Physical explanations of the vector triple product [p.9, l.2].
4. There is no displacement \( u \) on the right-hand side [p.10, l.12–l.11].

The formal operations given in [p.9, l.4; (2); p.12, l.1] make it difficult to see the key point.

Example 6.26. (Finding extrema with subsidiary conditions)

I. In calculus: The method of Lagrange multipliers [Reif [84, §A.10]].
II. In calculus of variations (usually consider minima): The analog of the method of Lagrange multipliers [Fomin–Gelfand [35, p.43, Theorem 1]].
III. In statistical mechanics [Reif [84, §6.8; §6.10]] (usually consider sharp maxima [Reif [84, p.202, l.8]]):

1. The method of Lagrange multipliers [Reif [84, p.229, l.16–p.231, l.13]].
2. Using the statistical trick: a rapidly increasing function multiplied by a rapidly decreasing function will produce a sharp maximum [Reif [84, p.222, l.1–p.223, l.8]].
The sharp maximum is produced by Reif [84, p.110, (3·7·14) or p.242, (7·2·15)].
3. Using the \( \delta \)-function and its Fourier transform [Reif [84, p.223, l.9–p.225, l.15]].

Example 6.27. (Physics proofs vs. mathematics proofs)

A physics proof is usually intuitive; it shows how we discover a new formula. In contrast, a mathematics proof is usually abstract; it shows how we prove it rigorously.

Example. \( \lim_{R \to \infty} \int_{-R}^{R} \sin(gx) \frac{f(x)}{\pi x} = f(0). \)

A physics proof. The nodal separation \( \pi/g \) of \( \sin(gx) \) becomes smaller and smaller as \( g \) becomes larger and larger. A small neighborhood \( (x - \pi/g, x + \pi/g) \) of \( x \neq 0 \) contributes to the integral a period of \( \sin(gx)/f(x)/x \), which is 0, where \( f(x)/x \) can be treated as a constant. In a neighborhood of \( x = 0 \), \( f(x) \) can be considered a constant. \( \int_{-\infty}^{\infty} \frac{\sin(gx)}{\pi x} f(0) = f(0) \) follows from Rudin [88, p.244, (7)].

A mathematics proof. The formula \( \int_{-\infty}^{\infty} \frac{\sin(gx)}{\pi x} f(x) = f(0) \) can be proved using an argument similar to the one given in Rudin [88, pp.243–244, Problem 10.44].

Example 6.28. (Distribution theory is a new theory that we create to avoid the contradiction that the domain of a function contains a point whose function value cannot be defined)

The concept of \( \delta \) generalized function originates from physics and is considered a function for many years by physicists so that some physicists think that we may treat the generalized function as a function when introducing the concept of \( \delta \) generalized function, may treat it so until reaching the critical juncture, and then jump to the right track by treating it as a generalized function. Perhaps this way will allow us to avoid a contradiction. In fact, treating the generalized function as a function in the beginning has already planted the seeds of contradiction. A contradiction does not occur simply because we fail to forsee it at
that time. Actually, a contradiction must occur. The concept of generalized function is a more delicate mathematical concept than that of function. The traditional mathematical language for functions is too crude to clearly explain the concept of generalized function.

\[ \nabla^2 \left( \frac{1}{|x-x'|} \right) = -4\pi \delta(x-x') \] [Jackson \[55\], p.35, l.1–4, (1.31)].

The first proof. See Pathria \[76\], p.501, l.3–l.11].

Remark. The above argument has problems. See Redžić \[83\], p.2, Remark \[†\]. However, if we handle the singular point carefully, we can make the above argument work as in the second proof.


Remark. In order to satisfy the conditions given in Cohen-Tannoudji–Diu–Laloë \[23\], vol. 2, p.1477, l.1–p.1478, l.1], we may let \( g_\varepsilon(r) = 1/\varepsilon (|r| \leq \varepsilon) \). The second proof is not rigorous because we treat \( \delta \)-function as a function rather than a generalized function [Rudin \[87\], p.141, l.7–l.3]. We discover that behind this significant but seemingly contradictory argument, there is actually a rich, deep and refined theory. The new theory requires more delicate analysis, language and formulation so that its meaning would not be ambiguous. A generalized function may be a function or a function whose domain contains a point which does not have a well-defined function value. However, for any testing function, the generalized function must have a well-defined value. Thus, distribution theory is a new theory that we create to avoid the contradiction that the domain of a function contains a point whose function value cannot be defined. The convergence \[ \lim_{\varepsilon \to 0} \frac{\sin(x/\varepsilon)}{\pi x} \to \delta(x) \] [Cohen-Tannoudji–Diu–Laloë \[23\], vol. 2, p.1470, (10)] should not be interpreted as the one in pointwise sense. Otherwise, we will have a contradiction [Pathria \[76\], p.498, l.1–p.498, l.7]. If we interpreted the convergence as the one in distribution sense [Rudin \[87\], p.146, l.9], the previous contradiction will not occur. If we use the concept of generalized functions, the second proof actually shows that \( g_\varepsilon \to -4\pi \delta \) in the distribution sense [Rudin \[87\], p.146, l.9]. Thus, the second proof can be made rigorous using distribution theory, so can the third proof.

The third proof. See Jackson \[55\], p.35. l.1–l.4].

Remark 1. Jackson’s proof is correct, but is disorganized because it uses the methods of distribution theory, but fails to use the theory’s terminology. A theory has its structures. Only through the use of the theory’s terminology may we clarify the structure of the proof and preserve its logical rigor.

Remark 2. The above proof can be translated into the language of distribution theory as follows: Let \( r_0 = \sqrt{r^2 + a^2} \).

\[
\begin{align*}
\lim_{a \to 0} < \nabla^2 \left( \frac{1}{r_0} \right) |\rho > &= \lim_{a \to 0} \int \int \int_{|r| \leq R} \rho(x) d^3x 
\lim_{a \to 0} \nabla^2 \left( \frac{1}{r_0} \right) \rho(x) \\
&= 4\pi \varepsilon_0 \Phi_0(x) = -4\pi \rho(0) = -4\pi \delta^{(3)}(0) |\rho >. \\
\text{Thus, } \lim_{a \to 0} \nabla^2 \left( \frac{1}{r_0} \right) &= -4\pi \delta_0 [\text{Rudin } \[87\], p.146, l.1–l.3].
\end{align*}
\]

The fourth proof. See Redžić \[83\], p.5, l.6–p.6, l.9].

Example 6.29. (How we deal with a problem that may easily cause us to commit errors)

To prove the equality given in Courant–John \[25\], vol. 2, p.568, l.1–p.568, l.11] may easily cause us to commit errors. Even worse, the situation is too confusing to allow us to locate errors. Is it because reality
often goes against mathematical conventions? If so, how should we prevent an error? If we commit an error, how should we find it and then correct it?

The advantage of the method given in Courant–John [25] over the direct calculation is that we need not carry out the somewhat complicated calculation of the second of \( u \) [Courant–John [25] vol. 2, p.567, l.20–l.22]. However, proving the equality given in Courant–John [25] vol. 2, p.568, l.12–l.11 may easily cause one to commit errors unless one is familiar with the consequences of choosing an orientation for a curve.

Define \( R_n \) as in Courant–John [25] vol. 2, p.567, l.6–l.4]. Let the polar coordinates of \( A, B, C, \) and \( D \) be \((r + h, \theta), (r + h, \theta + k), (r, \theta + k), \) and \((r, \theta)\).

I. The first parameterization \( \gamma_1 \) of \( C_n \):

Let \( s_0 < s_4 \). Define \( \gamma_1 : [s_0, s_4] \rightarrow C_n \) such that \( \gamma_1(s_0) = \gamma_1(s_4) = A, \gamma_1(s_1) = B, \gamma_1(s_2) = C, \) and \( \gamma_1(s_3) = D \).

II. The second parameterization \( \gamma_2 \) of \( C_n \):

Define \( \gamma_2 \) such that \( \gamma_2 : [\theta, \theta + k] \rightarrow AB, \gamma_2 : [r + h, r] \rightarrow BC, \gamma_2 : [\theta + k, \theta] \rightarrow CD, \gamma_2 : [r, r + h] \rightarrow DA \). The four segments of \( C_n \) are parameterized respectively as four different functions; it does not matter even if their domains intersect.

According to convention, the domain \([a, b]\) of a curve must satisfy \( a < b \). \( \gamma_2 : [r + h, r] \rightarrow BC \) does not comply with this convention. In fact, it reverses the sense of line segment \( \gamma_1(s_1, s_2) \) [Courant–John [25] vol. 1, p.334, l.19]). Since the principal normal is defined as the turning direction of the tangent vector, the principal normal of a point on \( \gamma_2[r + h, r] \) is the opposite of the principal normal of the corresponding point on \( \gamma_1(s_1, s_2) \).

III. The third parameterization \( \gamma_3 \) of \( C_n \):

In such a case, the key to preventing errors is to preserve the sense of the parameterization \( \gamma_1 \) when parameterizing \( C_n \). In order to fulfill this goal, all we have to do is reverse the orientation of each of the domains of the two segments of the parameterization \( \gamma_2 : [r + h, h] \rightarrow BC, \gamma_2 : [\theta + k, \theta] \rightarrow CD \). The rest of the segments of the parameterization \( \gamma_2 \) remain the same. The parameterization so formed is called \( \gamma_3 \).

Based on the parameterization \( \gamma_3 \), we may easily prove the equality given in Courant–John [25] vol. 2, p.568, l.12–l.11]. This is because a line integral is invariant under parameter changes if the orientation of the domain of a curve is preserved and also because the only segments of \( \gamma_3 \) whose principal normals do not point away from the origin or away from the polar axis are \( \gamma_3 : [\theta, \theta + k] \rightarrow CD \) and \( \gamma_3 : [r, r + h] \rightarrow DA \). Since \( du = \nabla u \cdot n \), the principal normals pointing toward the origin and those pointing toward the polar axis contribute the two minus signs in the formula. If we unfortunately choose the parameterization \( \gamma_2 \), we can still make it right as long as we pay attention to the above-mentioned remark about \( \gamma_2 \).

**Example 6.30.** (Generalized orientations)

When studying a generalized definition, we should understand its primitive version, its entire process of revolution, and the reason for the necessity of generalization. If we proceed directly toward the most general version in axiomatic approaches, its setting usually requires a more strange language and less familiar structures which may blur the essential idea, and the algorithm to check the definition usually becomes less effective. Thus, an improper approach to generalized definitions may easily lead to an empty formality and make it difficult for us to see the advantages of generalized definitions over the primitive version. Providing several non-trivial examples alone is not enough.

I. The approach given in Courant–John [25] vol. 2 aims at the origin, the insight, and the essential idea.

1. Choosing an advantageous setting makes us easily see the entire process of revolution [Courant–John [25] vol. 2, p.575, l.12–l.8].

2. Two ordered sets of vectors in \( \mathbb{R}^n \), \( (A_1, \ldots, A_n) \) and \( (B_1, \ldots, B_n) \), have the same orientation if and only if \( [A_1, \ldots, A_n; B_1, \ldots, B_n] > 0 \) [Courant–John [25] vol. 2, p.196, l.3–l.11].

3. Two ordered pairs of independent vectors on the tangent plane of a surface, \( (\xi, \eta) \) and \( (\xi', \eta') \), have the
same orientation if and only if \([\xi, \eta; \xi', \eta'] > 0\) [Courant–John [25] vol. 2, p.577, (40b)].


(5). \(S^*\) has the same orientation with respect to two ordered pairs of parameters \((u, v)\) and \((u', v')\) provided \(\frac{d(u, v')}{d(u, v)} > 0\) [Courant–John [25] vol. 2, p.581, (40s)].

(6). We use (5) [Courant–John [25] vol. 2, p.586, (41e)] instead of the positive unit normal to generalize the concept of orientation for a surface because on a manifold in higher dimensions there is no unique normal vector or “side” of \(S\) we can associate with \(S\) [Courant–John [25] vol. 2, p.583, 1.−7–p.585, 1.1]].

II. In contrast, although the orientation preserving or reversing for a vector space automorphism given in Spivak [95] vol. 1, p.114, 1.−5] is defined the same way as I (2), the setting for defining the concept of orientation is a non-trivial n-plane bundle [Spivak [95] vol. 1, p.116, 1.−3]], a generalization of tangent bundle. The unfamiliar setting and the direct axiomatic approach to the most general version may blur the essential idea. Therefore, the approach given in Spivak [95] vol. 1, p.114, 1.−8–p.118, 1.−7] is definitely not suitable for beginners even though providing several non-trivial examples in the end is still not good enough.

### Example 6.31.

(Using formulas in a table without care may easily result in mistakes)

\[
\frac{1}{\pi} \int_0^\pi \frac{a \, d\theta}{a - ib \cos \theta} = \frac{1}{\sqrt{a^2 + b^2}}, \quad \text{where the value of the square root is taken which makes} \quad |a + \sqrt{a^2 + b^2}| > |b| \quad \text{[Watson [109] p.384, 1.6; 1.12–l.13]].}
\]

**Proof.** Let \(z = e^{i\theta}\). Then

\[
a - ib \cos \theta = a - \frac{ib}{2} (z + \frac{1}{z}) = a - \frac{ib}{2} z = 2a z - ib^2 z.
\]

\[
\int_0^\pi \frac{a \, d\theta}{a - ib \cos \theta} = \frac{1}{2} \int_0^{2\pi} \frac{d\theta}{a - ib \cos \theta}. \quad \text{[Let} \quad f(\theta) = \frac{1}{a - ib \cos \theta}. \text{Then} \quad f(\pi - \theta) = f(\pi + \theta) \Rightarrow f - f = f(\pi + \theta) \text{d} \theta = f - f = \int_0^{\pi} \frac{f(\pi + \theta) \text{d} \theta}{\pi}]
\]

\[
= \frac{\pi}{b} \int_{|z|-1}^{\alpha} \frac{dz}{(|z| - \alpha)(|z| - \beta)} \quad \text{[where} \quad \alpha, \beta = i(-\frac{a}{b} \pm \sqrt{1 + \frac{a^2}{b^2}})]
\]

\[
= \frac{a + \sqrt{a^2 + b^2}}{a + \sqrt{b^2}} \quad \text{[by the residue theorem if we take the value of the square root such that} \quad |\alpha| = |\frac{a - \sqrt{a^2 + b^2}}{b} | < 1, \text{or equivalently} \quad |a + \sqrt{a^2 + b^2}| > |b| \text{since} \quad (a + \sqrt{a^2 + b^2})(a - \sqrt{a^2 + b^2}) = -b^2].
\]

**Remark 1.** The above argument is based on Conway [24] p.112, Example 2.9.

**Remark 2.** Using formulas in a table without care may easily result in mistakes. One is under the impression that once the solution form is obtained, the actual solution is determined. This is not so. If the resulting function is multivalued and the formula fails to indicate which value to choose, then the formula would be useless. One should find a delicate method to determine the correct value. If one uses such a unfinished formula in a proof, then the proof would be incorrect. Such a mistake is often difficult to detect. Here are some examples. Gradshteyn–Ryzhik [44] formula 6.611.1 fails to indicate which value of the square root to choose in order to get the correct answer. Because \((1 + z)^{-1/2}\) is a multivalued function, without assigning a specific value to \((1 + z)^{-1/2}\), it would be incorrect to prove \(\frac{1}{a} F\left(\frac{1}{2}, 1, 1, -\frac{b^2}{a}\right) = \frac{1}{\sqrt{a^2 + b^2}} \quad \text{[Guo–Wang [46 p.403, (3)]]}\) by using \((1 + z)^a = F(-\alpha, \beta, \beta, -z) \quad \text{[Guo–Wang [46 p.137, (10)]]}\). For example, one can make \((1 + z)^{-1/2}\) a single-valued function by defining it as in the binormal theorem. However, one has to pay a price for doing so. For example, there are two methods to calculate \(\int_0^\infty e^{-at}J_0(bt)\text{d}t\): one is using the binormal theorem to calculate \(\frac{1}{a} (1 + \frac{b^2}{a})^{-1/2}\) (one cannot calculate the square root in any other way) [Guo–Wang [46 p.403, 1.11, (3)]]; the other is interpreting the square root in the answer \(\frac{1}{\sqrt{a^2 + b^2}} \quad \text{[Watson [109 p.384, 1.6; 1.12–1.13].}}

---

67
Example 6.32. (How we detect errors in a textbook)

The formula given in Watson [109, p.388, (6)] and Guo–Wang [46, p.442, 1.3] should have corrected as

\[
\int_0^\infty e^{-\cosh \alpha} I_v(t) t^{\mu - 1} dt = e^{-\left(\mu - 1/2\right) \pi i} \sqrt{2 \pi \frac{\mathcal{Q}_{\nu-1}^{\mu-1}(\cosh \alpha)}{\sinh \alpha}} t^{\mu - 1/2} (\ast).
\]

How do we detect errors in a textbook? When I find an error, the first response is usually to refuse to accept this fact and try to rationalize the opposite viewpoint. After all, there are many authors who have not found it incorrect after copying it. For example, if we replace the factor \(e^{-\left(\mu - 1/2\right) \pi i}\) in (\ast) with \(\frac{\cos \nu \pi}{\sin (\mu + \nu) \pi}\), then we must consider \(e^{-2\pi i \mu} = 1\) true. Consequently, I try to rationalize this consequence: If I properly choose the value of \(\log(e^{-2\pi i})\), then \(e^{-2\pi i \mu}\) can be 1. Nevertheless, I try to remember this odd experience so that I can easily find a reason when a problem occurs afterwards. However, this “rationalization” actually conceals a mistake because \(e^{-2\pi i \mu} \neq 1\) if we let \(\mu = 1/2\). Thus, the reason why we fail to detect an error is that we have not gone far enough to forsee its consequences. Errors cannot withstand tests. Soon or later they will be detected. Even if an error may not be detected at the first checkpoint in application, it can hardly survive at the second one. When I tried to use Watson [109, p.388, (6)] and Guo–Wang [46, p.259, (4)] to prove Watson [109, p.388, (7)], I found that the coefficient of \(P_{1/2-\mu}^{1/2-\mu}\) supposed to be nonzero becomes 0 and the coefficient of \(P_{1/2-\mu}^{1/2-\mu}\) supposed to be 0 becomes very complicated if we consider \(\sin(\mu + \nu) \pi = \sin(\mu - \nu) \pi\) (a consequence of \(e^{-2\pi i \mu} = 1\)) true. Thus, the world would fall into pieces as if Pandora’s box were opened. I became so frustrated that I had to choose the other option: \(e^{-2\pi i \mu} = 1\) is not necessarily true. Then I found the counterexample: Case \(\mu = 1/2\). I could omit the story of proving Watson [109, p.388, (7)] and still make this paragraph logical, but this would destroy the evidence of true experience and eliminate the track of the natural thought for solving a problem.

Proof of (\ast). (A). (Whipple’s formula) \(e^{-\mu \pi i} Q_v^\mu(\cosh \alpha) = \sqrt{\frac{\pi}{2}} \frac{\Gamma(\mu v + 1)}{\sqrt{\sinh \alpha}} P_{v-\mu}^{-1/2}(\coth \alpha)\).

Proof. I. Let \(z = \cosh \alpha; y = P_{v-\mu}^{-1/2}(w)\), where \(w = \frac{z}{(z^2-1)^{1/2}}\). Then

\[
(1 - z^2) \frac{d}{dz} - 2z \frac{dz}{dz} + (y(v+1) - \mu^2) \frac{dy}{dz} = \left(z^2 - 1\right)^{-5/4} \left((1-w^2) \frac{d}{dw} - 2w \frac{dw}{dz} + (\mu^2 - \frac{1}{4}) - \frac{v+1/2}{1-w^2}\right) = 0.
\]

II. By I, \(u(z) = A Q_v^\mu(z) + BP_v^\mu(z)\). By Guo–Wang [46, p.249, (8); p.254, (4)], we have \(B = 0\) if we let \(v\) satisfy \(\Gamma(v + \frac{3}{2}) = \infty\).

III. \(\frac{Q_v^\mu(z)}{P_v^\mu(z)} \rightarrow A = e^{\mu \pi i} \Gamma(v + \mu + 1) \sqrt{\frac{\pi}{2}}\) as \(x \rightarrow +\infty\).

(B). Let \(\cosh \alpha = \coth \beta\). Then \(\sinh \alpha = \csch \beta\).

\[
\int_0^\infty e^{-\cosh \beta} I_v(t) t^{\mu - 1} dt = \frac{\Gamma(\mu + \nu)}{\sinh \beta} P_{v-\mu}^{-1/2}(\coth \beta) \text{ [Watson [109, p.387, (1)]; Guo–Wang [46, p.249, (9)]].}
\]

The result follows from (A).
analysis [Rudin 88, p.217, (1)]. To parameterize the integral contour for special functions, we often choose the argument of the integral’s dummy variable as the parameter. Once we choose a point on the contour and assign any possible value to its argument, then the value of the integral is determined by the direction of the contour. However, branch points frequently encountered in special functions may cause a lot of confusions and complications. In order to obtain the desired solution form and facilitate the calculations for the value of the integrand near a branch point, we must choose a proper point and assign a proper value to its argument.

II. Simple notations for complicated contour integrals: Watson–Whittaker [108, p.245, l.1] and 8–l.1–l.3.

III. Confusions and complications caused by branch points:

Watson [109, p.161, l.1–l.7–l.7] says, “We take the phases of \( t - 1 \) and \( t + 1 \) to vanish at the point \( A \).” Watson–Whittaker [108, p.257, l.1] says, “At the starting-point the arguments of \( t \) and \( 1 - t \) are both 0.” We may wonder if one variable with two conditions will cause a contradiction. Guo–Wang [46, p.353, l.1–l.7] says, “Assume \( |1 - t^2| = 0 \) along the path of integration.” One may wonder if this assumption is a prescription about which we should not question. Watson–Whittaker [108, p.257, l.1–l.15] shows that there are so many things concerning branch points to consider when we evaluate a contour integral. These confusions and complications are not what the authors intend to cause. The only purpose of [Watson–Whittaker [108, p.257, l.1–l.15; Watson [109, p.161, l.1–l.6]; Guo–Wang [46, p.353, l.1–l.7]] is to tell us that if we want to choose a point and its argument properly to facilitate calculations, we must consider branch points first.

IV. Convergence:

It is necessary to suppose that \( \Re(v + \frac{1}{2}) > 0 \) [Watson [109, p.161, l.1–l.10]]. This is because we must deal with branch points: the convergence of \( \int_{-1}^{1} (t + 1)^{\nu - \frac{1}{2}} dt \) or \( \int_{-1}^{1} (t - 1)^{\nu - \frac{1}{2}} dt \) requires \( \Re(v + \frac{1}{2}) > 0 \).

V. The advantage of representing special functions by contour integrals:

The two linearly independent solutions of the Bessel equation can be represented by the same integrand.

Example 6.34. (Tying up loose ends)

Both Watson [109, p.163, l.1–p.164, l.16] and Guo–Wang [46, p.355, l.1–p.356, l.1–l.1] prove Watson [109, p.164, l.3]. However, the ways they present have shortcomings. Let us tie up loose ends. Note that if we replace \( z = Re^{i\theta} \) in González [42] pp.680–681, Lemma 9.2 with \( z = Re^{-i\theta} \), then the lemma will become false. If the integrand is defined as in González [42] pp.680–681, Lemma 9.2, then integral along the path \( [0, \infty \exp(i\theta_1)] \) equals the integral along the path \( [0, \infty \exp(i\theta_2)] \) if \( \theta_1, \theta_2 \in [0, \pi] \). For this range of available half-lines, the positive real axis is the initial half-line; the positive imaginary axis is the middle half-line; the negative real axis is the final half-line. For the above reason, we may replace the integral path \( \int_{0}^{\infty} \) with \( \int_{0}^{\infty} \exp(i\theta) (0 \leq \theta \leq \pi) \) or replace \( \int_{0}^{\infty} \) with \( \int_{0}^{\infty} \exp(-i\theta) (|\theta| \leq \frac{\pi}{2}) \) if \( |\arg z| < \frac{\pi}{2} \) [Guo–Wang [46] p.356, l.1–l.2; Watson [109, p.164, l.4–l.14]]. Thus, the range of \( \theta \) depends on which reference half-line we choose. For the notation \( \int_{0}^{\infty} \exp(-i\omega) \), \( [0, \infty) \) represents our reference half-line and \( \omega \) is used to satisfy the condition \( |\arg z - \omega| < \frac{\pi}{2} \). By taking \( \theta_0 \in \{\omega| |\arg z| < \frac{\pi}{2} \} \cap \{\omega| |\arg z - \omega| < \frac{\pi}{2} \} \) \( \omega < \frac{\pi}{2} \), we may extend the domain of \( z \) from \( \{\omega| |\arg z| < \frac{\pi}{2} \} \cup \{\omega| |\arg z - \omega| < \frac{\pi}{2} \} \) [Watson [109, p.164, l.4–l.14]]. For example, if we let \( \theta_0 \rightarrow \) the positive imaginary axis and let \( \omega \rightarrow \frac{\pi}{2} \), we may extend the range of \( z \) from \( \{\omega| |\arg z| < \frac{\pi}{2} \} \) to \( \{\omega| |\arg z| < \frac{\pi}{2} \} \cup \{\omega| |\arg z - \omega| < \frac{\pi}{2} \} \) [Watson [109, p.164, l.4–l.14]]. In the former case, \( \int_{0}^{\infty} \exp(i\theta) = \int_{0}^{\infty} (-\frac{\pi}{2} < \theta \leq \pi) \); in the latter case, \( \int_{0}^{\infty} \exp(i\theta) = \int_{0}^{\infty} (-\pi \leq \theta \leq \frac{\pi}{2}) \). Thus, we will have more available half-lines along which the integrals equal the integral along the original positive imaginary axis.

Example 6.35. (The finishing touch)
Providing a solution to a problem alone is not enough; the author should tell the readers from where the solution comes. This way can bring the readers to an advantageous point for a bird's-eye view of the circumstance. By substitution,
\[(z-a)^α(z-b)^β(z-c)^γ\int_C(t-a)^β+γα-1(t-b)^γ+α+β'-1(t-c)^α+β+γ-1(z-t)-α-β-γ\,dt\] [Watson–Whittaker \[108\] p.292, l.14] is a solution of Riemann’s differential equation [Watson–Whittaker \[108\] p.283, l.1.13–1.16]. Watson–Whittaker \[108\] p.293, l.1.19–l.1.14] uses the definition of the beta function and the binomial theorem to prove that the given integral form is a solution, but Watson–Whittaker \[108\] §14-6] fails to explain from where the form comes. In contrast, Lebedev \[65, p.239, (9.1.3); (9.1.4)\] indicate that the integral form is built by means of the definition of beta function and the binomial theorem.

Example 6.36. (Musket to kill a butterfly)

The differentiation under the integral sign for \[\int_0^∞ \frac{t\,dt}{(z^2+t^2)(e^πt-1)}\] [Watson–Whittaker \[108\] p.250, l.1–4] \[\int_0^∞ \frac{\arctan(t/z)}{(z^2+t^2)(e^πt-1)}\,dt\] resp. [Watson–Whittaker \[108\] p.250, l.1–2] can be justified by either the classical method [Titchmarsh \[100\] p.100, l.7–l.13] or the modern method [Rudin \[88\] p.246, Exercise 16]. For the latter method, we let \(X = [0,∞), \mu = t(e^πt-1)^{-1}\,dt\), and \(\phi(z,t) = (z^2+t^2)^{-1}\) for \[\int_0^∞ \frac{tdt}{(z^2+t^2)(e^πt-1)}\]. Let \(X = [0,∞), \mu = t(e^πt-1)^{-1}\,dt\), and \(\phi(z,t) = \frac{\arctan(t/z)}{t}\) for \[\int_0^∞ \frac{\arctan(t/z)}{(z^2+t^2)(e^πt-1)}\,dt\]. From the hindsight, the uniform convergence of the infinite integral in Titchmarsh \[100\] p.100, l.12] hints the boundedness of \(\phi\) in Rudin \[88\] p.246, Exercise 16] for most cases.

Remark. Guo–Wang \[46, p.150, l.5–l.13\] or the modern method \[Rudin \[88\] p.246, Exercise 16\]. For the latter method, the differentiation under the integral sign is built. By substitution, the uniform convergence of the infinite integral in Titchmarsh \[100\] p.100, l.12] hints the boundedness of \(\phi\) in Rudin \[88\] p.246, Exercise 16] for most cases.

Example 6.37. (Grasping the overall situation)

M_{k,n}(z) = z^{1/2+n}e^{-z/2}\left\{1 + \frac{1/2+m-k}{11(2m+1)}z + \frac{(1/2-m-k)(3/2-m-k)}{2m+1(2m+2)}z^2 + \cdots\right\} and
M_{k,-m}(z) = z^{1/2-m}e^{-z/2}\left\{1 + \frac{1/2-m-k}{11(2m+1)}z + \frac{(1/2-m-k)(3/2-m-k)}{2m+1(2m+2)}z^2 + \cdots\right\} are two linearly independent solutions near \(z = 0\) of
\[
\frac{d^2W}{dz^2} + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{1/4-m^2}{z^2}\right\}W = 0 \quad [Watson–Whittaker \[108\] p.337, l.1.7–p.338, l.2].
\]

Proof. 1. \(F(α, γ, z)\) and \(z^{1−γ}F(α − γ + 1, 2 − γ, z)\) are two linearly independent solutions near \(z = 0\) of
\[ z^2 \frac{d^2 y}{dz^2} + (\gamma - z) \frac{dy}{dz} - \alpha y = 0 \] [Statement: Guo–Wang [46] p.297, (1), (2) & (3)]; proof: Lebedev [65] p.262, 1.7–p.263, 1.9].

II. By means of the transformation \( y = e^{z/2} z^{-\gamma/2} w(z) \),
\[ z^2 \frac{d^2 y}{dz^2} + (\gamma - z) \frac{dy}{dz} - \alpha y = 0 \]
is transformed to
\[ w'' + \left[-\frac{1}{4} + \frac{k}{z} + \frac{1}{4 - m^2 z^2} \right] w = 0 \]
i.e., \( w'' + \left[-\frac{1}{4} + \frac{k}{z} + \frac{1}{4 - m^2 z^2} \right] w = 0 \)
\( k = 2m \gamma / 2 - \alpha = k \).

\[ M_{k,m}(z) = e^{-z/2} z^{1/2+m} f(1/2 + m - k, 1 + 2m, z) \] [Guo–Wang [46] p.301, (5)]
\[ M_{k,-m}(z) = e^{-z/2} z^{1/2-m} f(1/2 - m - k, 1 - 2m, z) \] [Guo–Wang [46] p.301, (6)] are two linearly independent solutions near \( z = 0 \) of the last differential equation.

Remark 1. Watson–Whittaker [108] p.338, 1.3–l.10] gives a brief summary of the above proof. Watson–Whittaker [108] §16-1 shows that \( M_{k,m}(z) \) and \( M_{k,-m}(z) \) are solutions of Watson–Whittaker [108] p.337, (B) by substitution. This approach has the shortcoming of losing the beautiful structure of solution.

Remark 2. (Grasping the overall situation)

Hypergeometric functions and confluent hypergeometric functions are closely related. We must build paths between the two topics as many as possible. When we discuss confluent hypergeometric functions, of course, we have to include their characteristic properties. Furthermore, for each property, we should find its corresponding property in hypergeometric functions, treat the latter as a motivation of the former and use the latter to prove the former. Just because of the complicated circumstance, we should give a rigorous proof rather than touch it lightly. Otherwise, the discussion is incomplete.

Sneddon [91] p.32, 1.1–l.18] sets a good example for discussing confluent hypergeometric functions. It says that

I. By replacing \( x \) with \( x/\beta \) in Sneddon [91] p.23, (8.1)] (its formal solution is given by Watson–Whittaker [108] p.207, 1.7], \( F(\alpha, \beta; \gamma, x/\beta) \) is a solution of \( x(1 - \frac{x}{\beta}) \frac{d^2 y}{dx^2} + \left( \gamma - \left(1 + \frac{\alpha+1}{\beta}\right)x \right) \frac{dy}{dx} - \alpha y = 0 \).

II. By Hartman [48] pp.4–5, Theorem 2.4], \( \lim_{\beta \to \infty} F(\alpha, \beta; \gamma, x/\beta) \) is a solution of \[ x \frac{d^2 y}{dx^2} + (\gamma - x) \frac{dy}{dx} - \alpha y = 0. \]
Consequently, by the uniqueness of solution, we have

III. \( \lim_{\beta \to \infty} F(\alpha, \beta; \gamma, x/\beta) = \sum_{r=0}^{\infty} \frac{(\alpha)_{n/r}}{n!} x^n \).

By comparison, Lebedev [65] §9.9] only mentions III. However, its proof is incorrect: “a comparison of (9.1.2) and (9.1.3)” given in Lebedev [65] p.261, 1.11] should have been replaced with “a comparison of (9.1.6) and (9.1.1)”. Guo–Wang [46] §6.1] only mentions the transformation from hypergeometric equation to confluent hypergeometric equation [Guo–Wang [46] p.297, 1.1–l.17]. Therefore, both discussions are incomplete. Watson–Whittaker [108] chap. XVI] is poorly written because it is almost independent of Watson–Whittaker [108] chap. XIV]. A better of III is given as follows:

\textbf{Proof.} For \( 0 \leq n \leq t < n + 1 \), let \( f_{\beta}(t) = \frac{(\alpha)_{n/(\beta)})}{n!(\gamma)_{n}} \left( \frac{1}{\beta} \right)^n \), \( g(t) = \frac{(\alpha)_{n/(\gamma)})}{n!(\gamma)_{n}} \left( \frac{1}{\gamma} \right)^n \).
Let \( |x| \leq R \) and \( |\beta| \geq 2R \). Then
\[ f_{\beta}(t) \leq g(t) \] and \( \lim_{\beta \to \infty} f_{\beta}(t) = \frac{(\alpha)_{n}}{n!(\gamma)_{n}} x^n \)
\( (0 \leq n \leq t < n + 1) \).
Treat \( \sum_{n=0}^{\infty} \) as \( \int_{0}^{\infty} \) and apply Rudin [88] p.27, Theorem 1.34] to this case.

\[ F(\alpha, \gamma; z) = e^{z} F(\gamma - \alpha, \gamma; -z) \] [Guo–Wang [46] p.298, (6)] can be proved similarly.

\textbf{Proof.} \( F(\alpha, \beta; \gamma, \frac{z}{\beta}) = (1 - \frac{z}{\beta})^{1-\beta} F(\beta, \gamma - \alpha, \gamma; -\frac{z}{\beta}) \) [Guo–Wang [46] p.143, (10)].
For \( 0 \leq n \leq t < n + 1 \), let \( f_{\beta}(t) = \frac{(\beta)_{n}(\gamma - \alpha)_{n}}{n!(\gamma)_{n}} \left( \frac{z}{\gamma - \beta} \right)^n \) and \( g(t) = \frac{(\beta)_{n}(\gamma - \alpha)_{n}}{n!(\gamma)_{n}} \left( \frac{1}{\gamma} \right)^n \). Then
\(f_\beta(t) \leq g(t)\) if \(|z| \leq R\) and \(|\beta| \geq 2R\).

**Example 6.38.** (Linear transformations of the hypergeometric function)

I. By comparing Watson–Whittaker \([108] \S 14-3 \& \S 14-4\) with Guo–Wang \([46] \S 4.3\), we have the following results:

(A). The former considers the general equation of Fuchsian type having three regular singularities [Guo–Wang \([46] \text{p.68, (1)}\)], while the latter considers the standard hypergeometric equation [Watson–Whittaker \([108] \text{p.207, Example}\)]. It is sufficient for our purpose to consider the standard type. In addition, it is much simpler.

(B). The former lists 24 solutions first, and then keeps 6 of them by eliminating repetitions. Through this trial-and-error approach, Watson–Whittaker \([108] \S 14-4\) finally obtains three pairs of solutions, each pair corresponding to a regular singularity [Watson–Whittaker \([108] \text{p.206, l.17–l.16}\)]. The ineffective counting shows that we should redesign our counting plan to fit our needs. That is, we should use the correspondence between solution pairs and regular singularities as the guide to redesign our counting plan. This is exactly the approach of Guo–Wang \([46] \S 4.3\).

(C). Furthermore, Guo–Wang \([46] \text{p.141, (4) \& (5)}\) can be derived from Guo–Wang \([46] \text{p.140, (2) \& (3)}\) by inspection [Watson–Whittaker \([108] \text{p.207, (I) \& (II)}\)]. In contrast, if we express the solutions of the hypergeometric equation by Riemann’s P-equation [Lebedev \([65] \text{p.248, (9.5.4)}\)], then the solution of the transformed hypergeometric equation can be found by inspection [Watson–Whittaker \([108] \text{p.206, l.1–7}\)] to handle \([14] \cdot \text{Example 6.38.} \quad \text{(Methodical solutions)}

\[
W_{k,m}(z) = -\frac{1}{2\pi i} \Gamma(k + \frac{1}{2} - m) e^{-z/2} z^{k-1/2+m} \int_{\infty}^{(0+)} (-t)^{-k-1/2+m} (1 + \frac{t}{z})^{k-1/2+m} e^{-t} dt \quad [\text{Watson–Whittaker } [108]]
\]

Remark. Reversing the order of summation and integration can be justified by Rudin \([88] \text{p.150, Theorem 7.8}\) or Rudin \([88] \text{p.27, Theorem 1.34}\). By proving the above statement both ways, we may see the close relationship between the Fubini theorem and Lebesgue’s dominated convergence theorem.
The differential equation given in Watson–Whittaker \[108\] p.291, l.11–l.7] belongs to a special type. The given solution is justified simply by substitution [Watson–Whittaker \[108\] p.292, l.15–l.10]. We do not know from where the integrand comes. The underdeveloped solution based on guess, luck, and trial-and-error such as Watson–Whittaker \[108\] p.339, l.13–l.12] cannot be considered a methodical solution. In contrast, the integral solution given in Guo–Wang \[46\] p.305, l.10–l.19; §6.4] is built by a systematic method which applies to the wider class of equations of Laplacian type [Guo–Wang \[46\] §2.13]. In fact, the integrand and the path of integration [Guo–Wang \[46\] p.302, l.4–l.13] can be specified by the Laplace transform. Consequently, the latter solution is more methodical than the former one.

**Example 6.40.** (Applications of analytic continuation to the Weber–Schaafheitlin integral: the right timing for a statement’s appearance)

Suppose we choose the weakest possible conditions required in an argument to be our theorem’s hypothesis. If the argument has used the method of analytic continuation [Rudin \[88\] §16.9–§16.16] no more than once, then no confusion will occur. However, what should we do if the argument has used the method of analytic continuation more than once? Let us see the following example.

Example. Let \( A(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(z/2)^{2m+1}}{\Gamma(\alpha+2m+1)} \),
\[
B(z) = \int_{0}^{\infty} \frac{J_{\alpha}(zt)}{t^{\alpha}} dt, \quad D_1 = \{z | \Re(z) > 2a\};
\]
\[
C(z) = \frac{1}{2\pi i} \int_{C} \frac{(at)^{\alpha}}{(at)^{\alpha+2}} \frac{\Gamma(\alpha+2m+1)}{\Gamma(\alpha+2m+1)} \frac{(\alpha+2m+1)}{\Gamma(\alpha+2m+1)} (-s) ds, \quad D_2 = \{|z| \arg z < \pi\}.
\]

Watson \[109\] p.402, l.13–l.19 shows that \((B, D_1)\) is an analytic continuation of \(A\); Watson \[109\] p.402, l.10–l.4 shows that \((C, D_2)\) is analytic continuation of \(A\). Since \(D_1 \subset D_2\), we can say that \((B, D_2)\) is an analytic continuation of \(A\). In order to establish the first analytic continuation, we must impose the condition \(z \in D_1\). After establishing the second analytic continuation, we find that the condition \(z \in D_2\) can be weakened to the condition \(z \in D_2\). However, before we establish the second analytic continuation, there is no way to know that \((B, D_2)\) is an analytic continuation of \(A\). Thus, the paragraph given in Watson \[109\] p.402, l.13–l.11] has the problem with timing; we should collect enough evidence before we propose a hypothesis. Therefore, whenever we use the method of analytic continuation, we should check and record if the change of the condition is needed so that we may easily clarify the relationship between cause and effect in the proof structure.

In fact, Watson \[109\] §13.4; §13.41] are self-contained, but its author has written the facts in the form of previews because of the timing problem. Every time he says that a condition ensures convergence, the readers may not be able to prove the fact at that moment, but they should be able to find the proof later in the section if they are patient enough. However, some impatient readers may think that they must find the proof somewhere else. The incorrect claim given in Guo–Wang \[46\] p.405, l.7–l.9] is sufficient to show that there are many people under the mistaken impression. In fact, one cannot see the convergence of \( \int_{0}^{\infty} \frac{J_{\alpha}(at)}{t^{\alpha+1}} dt \) [Watson \[109\] p.399, l.12–l.15] until one reads up to Watson \[109\] p.401, l.15]. Similarly, one cannot see the convergence of \( \int_{0}^{\infty} \frac{J_{\alpha}(at)}{t^{\alpha+1}} dt(\mu - \nu \text{ is an odd integer; } 0 > \Re(\lambda) > -1) \) [Watson \[109\] p.403, l.8] until one reads up to Watson \[109\] p.404, (3)].

Remark 1. \( \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(z/2)^{2m+1}}{\Gamma(\alpha+2m+1)} \int_{0}^{\infty} e^{-ct} J_{\alpha-\beta}(at) t^{\alpha+\beta+2m-1} dt \) is absolutely convergent when \(|z| < c\) [Watson \[109\] p.399, l.9–l.8].

**Proof.** By Watson \[109\] p.385, (2)], \( \int_{0}^{\infty} e^{-ct} J_{\alpha-\beta}(at) t^{\alpha+\beta+2m-1} dt = O(e^{-2c(2m+1)}) \). Then use the ratio test.

\(\square\)
Remark 2. We impose the condition $\Re z > 0$ [Watson [109], p.399, 1.18] because $z^{r+2m+1}$ [Watson [109] p.399, 1.20] requires the consideration of the domain of $\log z$. We impose the condition $|\Im z| < c$ to ensure the convergence of $\int_0^\infty e^{-at}J_{r-\beta}(\alpha t)^{-\delta}dt$; see Jackson [55], p.114, (3.91). Let $D = \{z|\Re z > 0, |z| < c\}$, $D' = \{z|\Re z > 0, |z| < \sqrt{a^2 + c^2} - c\}$ and 

$$f(z) = \int_0^\infty e^{-at}J_{r-\beta}(\alpha t)^{-\delta}dt = \sum_{m=0}^{\infty} \frac{(-1)^m(\gamma/2)^{2m}}{m!\Gamma(m+1)}z^{\frac{2m}{\alpha}} \text{ } _2F_1(\alpha + m, 1/2 - \beta - m; \alpha - \beta + 1; \frac{\alpha}{\alpha^2 + c^2})$$

Watson [109], p.399, 1.14–1.5] shows that $f(z)$ is analytic on $D$. Watson [109] p.400, 1.1–1.23 shows that $(f,D')$ is an analytic continuation of $(f,D)$. In order to prove the analyticity of $f$ on $D$, we impose the condition $|z| < c$; after the establishment of the analytic continuation $(f,D')$, we find that the condition $|z| < c$ can be weakened to the condition $|z| < \sqrt{a^2 + c^2} - c$. Thus, using the method of analytic continuation is like having a higher floor: our views become broader and farther.

Remark 3. By Rudin [86] p.135, Theorem 7.11], the limit of the series when $c \to 0$ is the same as the value of the series when $c = 0$ [Watson [109], p.401, 1.8–1.9]. "Provided that the integral is convergent" [Watson [109], p.401, 1.6] means “the condition given in Watson [109], 1.3] is satisfied”.

Remark 4. By Jackson [55], p.114, (3.91), $J_{\alpha-\beta}(at), J_{\gamma-\beta}(at) = O(e^{at})$. In order to ensure the convergence of $\int_0^\infty e^{-at}J_{\alpha-\beta}(at)J_{\beta-\gamma}(at)dt$, we impose the condition $\Re z > 2a$ [Watson [109], p.402, 1.20].

Remark 5. If $\Re z > 0$ and $|z| < 2a$, then

$$\int_0^\infty e^{-at}J_{\alpha-\beta}(at)J_{\beta-\gamma}(at)dt = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m(\gamma/2)^{2m}}{m!\Gamma(m+1)}(\alpha^2 - \beta^2 - \gamma^2 + 2m^2 \Gamma(\beta+2)\Gamma(\gamma+2)\Gamma(\alpha^2 - \beta^2 - \gamma^2 + 2m^2)/\Gamma(\alpha^2 - \beta^2 - \gamma^2 + 2m^2))$$

\[Watson [109], p.403, 1.3–1.5].\]

**Proof.** We choose Watson–Whittaker [108], p.288, 1.16–p.289, 1.5] or Guo–Wang [46, p.154, Fig. 9] to be our primitive model for development. By Guo–Wang [46], p.100, (8)], $\Gamma(2a+2s)$ provides the factor $2^{2s}$, so does $\Gamma(\alpha - \beta + \gamma + 2s)$. The numerator of the integrand given in Watson [109], p.402, 1.8–1.14] provides the factor $(a/2)^{2s}$. Consequently, instead of considering $|(z^\alpha)\beta \csc \pi |$ [Guo–Wang [46], p.155, 1.10]], we should consider $|\frac{4\pi}{\zeta^2}2 \sin \pi \rho |$

$$= O(\exp[(N + 1/2) \cos \theta \ln \left| \frac{4\pi}{\zeta} \right|^2 - (N + 1/2) \delta \sin \theta]) \ln \left| \frac{2\pi}{\zeta} \right| > 0 \text{ because } |z| < 2a$$

\[O(\exp[-21/2(N + 1/2) \ln \left| \frac{4\pi}{\zeta} \right|^2]) \text{ if } -\pi < \theta \leq -3\pi/4 \text{ or } 3\pi/4 < \theta \leq \pi \]

\[O(\exp[-21/2(N + 1/2)]) \text{ if } -3\pi/4 < \theta \leq -\pi/2 \text{ or } \pi/2 < \theta \leq 3\pi/4. \]

Remark 6. $z^\gamma - \alpha - \beta$ $\Rightarrow e^{(\gamma - \alpha - \beta)\ln z}$$|z^\gamma - \alpha - \beta| = e^{|\gamma - \alpha - \beta|\ln |z| - \arg z} |\gamma - \alpha - \beta|.$

If $\Re(\gamma - \alpha - \beta) > 0$ and $z = c \to 0$, then $|z^\gamma - \alpha - \beta| = e^{\Re(\gamma - \alpha - \beta)\ln c} \to 0$ [Watson [109], p.403, 1.1]].

Remark 7. “It is supposed that these relations hold down to the end of §13.41.” [Watson [109], p.399, 1.12] should have been corrected as follows:

“In Watson [109], p.399, 1.7–p.403, 1.9], $(\mu, \nu, \lambda) \leftrightarrow (\alpha, \beta, \gamma)$ is transferred according to the relations given in Watson [109], p.399, 1.9–1.11]; $\alpha = (\mu + \nu - \lambda + 1)/2$. In Watson [109], p.403, 1.7–8–p.404, 1.7], $(\mu, \nu, \lambda) \leftrightarrow (\alpha, p, \lambda)$ is transferred according to the relations given in Watson [109], p.403, 1.6–1.5]; $\alpha = (\mu + \nu + 1)/2$.”

It is really confusing to use the same notation $\alpha$ in the same section [Watson [109], §13.41] to represent two different quantities. The latter $\alpha$ should have been replaced with another notation, for example, $\eta$.

Remark 8. Without loss of generality we may assume that $p = 0, 1, 2, \cdots$ [Watson [109], p.403, 1.8; 1.6–1.5].

Remark 9. Since $\Re \lambda < 0$, by Bromwich [16], p.203, 1.7–1.5; p.204, 1.17–1.15], both $\_2F_1(\alpha - \frac{1}{2}, \frac{1}{2} - p$
\(2: \alpha - p; 1\) and \(2F_1(\alpha - \frac{1}{2}, \alpha + p - \frac{1}{2}; \alpha + p + 1; 1)\) diverge [Watson [109] p.404, 1.10–1.11]. The following supplements may help us understand the proof of the theorem given in Bromwich [16] p.204, 1.13–1.22:

(1). In order to obtain \(\frac{a_n}{a_{n+1}} < 1 + \frac{\sigma}{n}\) [Bromwich [16] p.34, 1–8], we must impose the condition that \(\sigma_n\)'s are bounded.

(2). \(\sum a_n\) converges \(\leftrightarrow \lim(na_n) = 0\) [Bromwich [16] p.35, 1.16–1.17].

Proof. I. \(\frac{n a_n}{(n+1)a_{n+1}} = 1 + \frac{1}{n}(\mu - 1) \frac{n}{n+1} + \frac{n}{n+1} \frac{a_n}{a_{n+1}}\).

II. \(\Rightarrow\):

By Bromwich [16] p.35, 1.12, \(\mu > 1\).

By I and Bromwich [16] Art. 39, Ex. 3, \(\lim(na_n) = 0\).

\(\Leftarrow\):

By Bromwich [16] p.35, 1.12, \(\mu \leq 1\).

Case \(\mu < 1\): By I, \(\frac{n a_n}{(n+1)a_{n+1}} \leq 1\). Hence, \(na_n \nearrow\).

Case \(\mu = 1\): By induction, \(\sum a_m = O(na_n)\).

If \(\sum a_n\) diverges, then there exists an subsequence \(n_k\) such that \(\sum_{n=1}^{n_k} a_m \to L \neq 0\).

\(\exists M > 0\): \(|L(n_k a_n)| \leq M\). This contradicts \(\lim(na_n) = 0\).

(3). By Rudin [86] p.62, Theorem 3.43, the hypergeometric series given in Bromwich [16] p.35, 1–16] converges for \(x = -1\), if \(\gamma + 1 > \alpha + \beta\).

(4). Without imposing proper conditions, the three theorems given in Bromwich [16] p.201, 1.4; 1.5; 1–10] cannot be valid. However, our goal is proving the theorem given in Bromwich [16] p.204, 1.13–1.22]. Consequently, all we have to do is impose some conditions so that the above three theorems are valid for the cases (1), (2), and (3) given in Bromwich [16] p.204, 1–3]. For example, the theorem given in Bromwich [16] p.201, 1.5] is valid for case (3) because \(\frac{a_n}{a_{n+1}} + \frac{D_n}{D_{n+1}} > \frac{1}{2} (\frac{a_n}{a_{n+1}} \to 1 \text{ and } \frac{D_n}{D_{n+1}} \to 0 \text{ as } n \to \infty)\). The proof of \(\lim \kappa_n > 0\) [Bromwich [16] p.201, 1–10)] can be proved as follows:

Proof. \(\lim[\frac{f(n)}{f(n+1)} - \frac{f^2(n+1)}{f(n+1)}] > 0\) [Bromwich [16] p.201, 1.4].

\(f(n)^2 + 2f(n)f'(n) - f^2(n+1)\)

\(\equiv -2 \int_0^1 (f(n+x) - f(n)) f'(n) dx\)

\(\equiv -2 \int_0^1 \frac{d}{dt} f(n+t) f'(n+t) dt\)

\(\equiv -2 \int_0^1 f(n+t) (f'(n+t) + f''(n+t)) dt\).

For cases (1), (2), and (3), \(\frac{f(n+t)}{f(n)} \leq 1\). \(f''(n+t), f'(n+t) \to 0\) [Bromwich [16] p.201, 1–5] as \(n \to \infty\).

(5). \(\lim(\log n) n\{\frac{a_n}{a_{n+1}}^2 - 1\} - 2\} > 2\) (convergence); \(\lim(\log n) n\{\frac{a_n}{a_{n+1}}^2 - 1\} - 2\} < 2\) (divergence) [Bromwich [16] p.202, 1.5, 3)] should have been corrected as \(\lim(\log n) n\{\frac{a_n}{a_{n+1}}^2 - 1\} - 2\} \to 0\) (convergence); \(\lim(\log n) n\{\frac{a_n}{a_{n+1}}^2 - 1\} - 2\} < 0\) (divergence).

(6). If \(\alpha = 0\), then \(|a_m| \to L > 0\) as \(m \to \infty\) [Bromwich [16] p.203, 1–3–1.5].

Proof. \(|\frac{a_n}{a_{n+1}}^2 = 1 + \frac{\sigma}{n^2}\).

\(1 - \varepsilon n^{\delta - \lambda} \leq |\frac{a_n}{a_{n+1}}|^2 \leq 1 + \varepsilon n^{\delta - \lambda}\). Consequently,

\(1 - \varepsilon \sum_{k=n}^{m} k^{\delta - \lambda} \leq \sum_{k=n}^{m} |\frac{a_n}{a_{n+1}}|^2\) [Bromwich [16] p.95, 1–9]

\(\leq 1 + \varepsilon \sum_{k=n}^{m} k^{\delta - \lambda}\).
Remark 11. By Watson \cite[p.403, (2)]{Watson}, \( \sum \). Assume that

\begin{equation}
(\text{Integration on a Riemann surface with branch points})
\end{equation}

\[
\int \alpha \text{d}x
\]

\[
\text{prove that}
\]

\[
\text{For example, in order to prove Watson \cite[p.168, (3)]{Watson}, we must}
\]

\[
\text{often have to degenerate a part of the contour to a point. In order to make the}
\]

\[
\text{argument of points along the}
\]

\[
\text{the latter integral may depend on which sheet the line segment is in, while the former integral is an invariant}
\]

\[
\text{quantity. When we reduce a contour integral on a Riemann surface to an integral along a line segment, we}
\]

\[
\text{often have to degenerate a part of the contour to a point. In order to make the argument of points along the}
\]

\[
\text{contour continuous and simplify the calculation of these arguments, we should restore the degenerated point}
\]

\[
\text{to its corresponding nondegenerate part. For example, in order to prove Watson \cite[p.168, (3)]{Watson}, we must}
\]

\[
\text{prove that}
\]

\[
\text{Let } I = \int \text{exp} \text{e} \text{u} = \text{d}u
\]

\[
\text{IA and CI be line segments; AB and BC are}
\]

\[
\text{counterclockwise half-circles.}
\]

\[
\text{IA and CI are on different sheets.}
\]

\[
\text{We take the argument of } -u \text{ in the range between } \beta - 2\pi \text{ and } \beta.
\]

\[
\int_{BC} = \int_{AB} + \int_{BCI}
\]

\[
\text{The ending point of the integration path } [\text{exp} \text{e} \text{u}, 0] \text{ comes from the ending point of the integration path}
\]

\[
\text{IAB, namely, } B. \text{ So the argument of } u \text{ at the } u = 0 \text{ is } \beta - \pi. \text{ Then the argument of } -u \text{ at the } u = 0 \text{ is } \beta. \text{ Thus,}
\]

\[
\beta - (\beta - \pi) = \pi.
\]

**Proof.** Let \( I = 1 + io, A = 1 + \delta e^{-3\pi i/2}, B = 1 + \delta e^{-\pi i/2}, C = 1 + \delta e^{-\pi i/2}; IA \) and \( BC \) be counterclockwise half-circles.

\[
\int_{\gamma}^{(0)} = \int_{IA} + \int_{BC}.
\]

Based on the restriction given in Guo–Wang [46, p.371, l.10], at the beginning point of integration path, the argument of \( (\pi - 2) \) is \(-\pi/2\), while the argument of \( \pi - 2 \) is \(-\pi/2\), so

\[
\int_{IA} = \int_{1+i\delta}^{\infty} e^{iz} (t^2 - 1)^{1/2} dt = (e^{i\pi})^{1/2} \int_{1+i\delta}^{\infty} e^{iz} (1 - t^2)^{1/2} dt.
\]

\[
\int_{BC} = \int_{1+i\delta}^{\infty} e^{iz} (t^2 - 1)^{1/2} dt = -\int_{1+i\delta}^{\infty} e^{iz} (1 - t^2)^{1/2} dt
\]

\[
= -(e^{i\pi})^{1/2} \int_{1+i\delta}^{\infty} e^{iz} (1 - t^2)^{1/2} dt.
\]

This is because at the beginning point of the integration path, the argument of \( t - 1 \) is \( \pi/2 \), while the argument of \( 1 - t \) is \(-\pi/2\).

\( \square \)

**Remark.** The above proof shows that \( \int_{IA} \) is itself the integral for \( S \). We may assume that \( \arg(1) = \pi/2 \) and \( \arg(-1) = 0 \).

**Example 6.42.** (Contour integrals for Bessel functions)

Example 1. \( S_{\nu, \alpha, \beta, \gamma}(\rho, t; a) = \int_{0}^{\infty} J_{\nu}(\rho x) J_{\gamma}(tx) x^{\nu+1} dx + \frac{2}{\pi} \sin \left( \frac{\alpha + \beta + \gamma - 2\nu}{2} \right) \nu K_{\nu, \alpha, \beta, \gamma}(\rho, t; a) \) [Sneddon 93, p.35, 1.15–1.16, (2.2.9)].

**Proof.** I. Let \( C_R = \{Re^{i\theta} | 0 \leq \theta \leq \frac{\pi}{2} \} \) and \( F(z) = \frac{J_{\nu}(az) + iY_{\nu}(az)}{K_{\nu}(az)} I_{\alpha}(\rho z) J_{\gamma}(t z) z^{1+\gamma} \).

We want to prove \( \lim_{R \to \infty} \int_{C_R} F(z) dz = 0 \).

**Proof.** \( \frac{1}{\pi R} e^{i\Re(iaz)} \) and \( |J_{\alpha}(\rho z)| \leq \frac{2}{\pi R} e^{i\Re(\rho z)} \).

We may assume that \( \arg(iz) \) lies between \( \frac{\pi}{2} \) and \( \pi \).

\( \square \)

II. \( \int_{0}^{\infty} F(z) dz = -\int_{0}^{\infty} J_{\nu}(a iy) + i Y_{\nu}(a iy) \frac{J_{\alpha}(\rho iy) J_{\gamma}(t iy) (iy)^{1+\gamma} d(iy).}

J_{\nu}(a iy) + i Y_{\nu}(a iy) = H_{\nu}^{(1)}(a iy) \) [Watson 109, p.73, (1)]

\[= \frac{2}{\pi} K_{\nu}(ay) i^{\nu-1} \] [Jackson 55, p.116, (3.101)].

J_{\nu}(a iy) = i^{\nu} K_{\nu}(ay) \) [Jackson 55, p.116, (3.100)].

\(-\Re([i^{-2\nu} + a + \beta + \gamma] + 1) = -\Re([e^{\pi i/2(-2\nu + a + \beta + \gamma)]}

\(-\Re([\cos((-2\nu + a + \beta + \gamma)/2) + i \sin((-2\nu + a + \beta + \gamma)/2)] i)

= \sin((-2\nu + a + \beta + \gamma)/2).

III. Let \( \Gamma = \sum_{s=1}^{p} Y_{s} \). Then

\[\int_{\Gamma} F(z) dz = -\pi i \sum_{s=1}^{p} \text{Res} F(\lambda_{s}) \] [González 42, p.683, Lemma 9.4].

Y_{\nu}(a \lambda_{s}) = \frac{-a^{2}}{\pi a \lambda_{s} K_{\nu}^{\prime}(a \lambda_{s})} \) [Watson 109, p.76, 1.2–1.3]

\[= \frac{2}{\pi a \lambda_{s} K_{\nu}^{\prime}(a \lambda_{s})} \] [Watson 109, p.45, (4)].
\[ \lim_{p \to \infty} \int F(z) dz = -S_{v,\alpha,\beta,\gamma}(\rho,\tau;\alpha) \text{ [Guo–Wang \([46]\ p.422, 1.4–1.10)]}. \]

IV. The desired result follows from Watson \([109]\ p.482, 1.4–1.5\) and Cauchy’s theorem.

Example 2. \(S_{v,\alpha,\beta,\gamma}(\rho,\tau;\alpha)\) be derived from Watson \([109]\ p.482, 1.4–1.5\] and Cauchy’s theorem.

Proof. I. Let \(C_R = \{Re^\theta | 0 \leq \theta \leq \frac{\pi}{2}\} \) and \(F(z) = \phi(z)\int F(z) dz = 0\), where
\[ \phi(z) = \frac{\left(zJ'_{v}(z)+HJ_{v}(z)\right)+H\left(J'_{v}(z)+HJ_{v}(z)\right)}{2\pi i} \]
Then \(\lim_{R \to \infty} \int_{C_R} F(z) dz = 0\).

II. \(\int_{|z|=1} \Theta F(z) dz = 2\pi i \sum_{n=1}^{p} \text{Res } F(\mu_n) \text{ [González \([42]\ p.683, Lemma 9.4)]}.
\[ \int_{|z|=1} \Theta F(z) dz = 2\sum_{n=1}^{p} \left(\frac{J_{\mu_n}(\mu_n)J_{\mu_n}(\mu_n)}{\mu_n - \nu^2 + H^2} \right) \mu_n^2 + \delta (0 < u, 1 < v < 1) \text{ [Watson \([109]\ p.480, 1.21–1.24)]}. \]

Proof. \(\frac{d}{dz} \left(\frac{zJ'_{v}(z)+HJ_{v}(z)}{J'_{v}(z)+HJ_{v}(z)}\right) = -\frac{H^2 + \mu^2 + \nu^2}{\mu_n} \frac{J_{\mu_n}(\mu_n)}{J_{v}(\mu_n)} \text{ [Watson \([109]\ p.76, 1.19, 1.20)]}. \]

IV. The desired result follows from Watson \([109]\ p.482, 1.21–1.19\) [we must assume that \(v \geq H\)] and Cauchy’s theorem.

Example 6.43. (The recurrence formulas for Neumann’s polynomials)

The recurrence formulas for Neumann’s polynomials given by Watson \([109]\ p.274, 1.19, 1.20\) can be derived from

I. The relation between Bessel and Neumann’s polynomials: Watson \([109]\ p.271, 1.19, 1.20\).

II. The Laurent series expansion for the generating function of Bessel coefficients: Watson \([109]\ p.14, 1.19, 1.20] \text{ and } \text{Watson } \([109]\ p.14, 1.19, 1.20].

III. The recurrence formula for Neumann coefficients: Watson \([109]\ p.45, 1.19, 1.20\] (see Watson \([109]\ p.275, 1.19, 1.20]).

Remark 1. (Want to prove uniform convergence when convergence is given)
\[ \sum a_n \left(\frac{z}{t}\right)^n \text{ converges uniformly in } (z, t) \iff \sum a_n J_n(z)O_n(t) \text{ converges uniformly in } (z, t) \text{ [Watson \([109]\ p.274, 1.3–1.5)]}. \]

Proof. Since \(|z| \leq (\limsup \nrightarrow 4\sqrt{|a_n|})^{-1}, \sum \frac{a_n}{n!}O(n^{-2}) \text{ converges uniformly in } (z, t). \]

\(\Rightarrow:\) The desired result follows from Watson \([109]\ p.273, 1.19, 1.20]\ and Bromwich \([16]\ p.113, 1.19, 1.20].

\(\Leftarrow:\) We must assume that \(|t|\) is bounded.
\[ \sum \frac{a_n}{n!} \left(1 - \frac{z^2}{4n^2}\right) \text{ converges uniformly in } (z, t). \]

\[ \left|\sum_{m=0}^{m+p} \frac{a_n}{n!}\right| \leq t^2 \left[1 + (\limsup \nrightarrow 4\sqrt{|a_n|})^{-2}\right] \sum_{m=0}^{m+p} \frac{a_n}{n!} \text{ [Bromwich \([16]\ p.113, 1.19, 1.20)]}. \]
Remark 2. (Series rearrangement) \( J_0(z)[O_0(t) - tO_1(t)] + J_1(z)[2O_1(t) - \frac{1}{2}tO_2(t) + \frac{1}{2}] + \sum_{n=2}^{\infty} J_n(z)[2O_n(t) - \frac{iO_{n+1}(t)}{n+1} - \frac{tO_{n-1}(t)}{n-1} + \frac{2n\sin(n\pi/2)}{n^2-1}] = 0 \) [Watson 109, p.275, 1.9–1.10].

Proof. \( \sum_{n=1}^{m} [J_{n+1}(z) + J_{n+1}(z)] \cdot tO_n(t) = \frac{J_0(z)[O_0(t) - tO_1(t)] + J_1(z)[2O_1(t) - \frac{1}{2}tO_2(t) + \frac{1}{2}]}{n} + \sum_{n=2}^{m} J_n(z) \left[ \frac{iO_{n+1}(t)}{n+1} + \frac{tO_{n-1}(t)}{n-1} - \frac{2n\sin(n\pi/2)}{n^2-1} \right] + J_m(z) \left[ tO_m(t) - \cos^2[(m-1)\pi/2]/(m-1) + J_m(z)[tO_m(t) - \cos^2(m\pi/2)]/m. \right. □

Remark 3. (Detailed analysis) The statements given in Watson 109, p.275, 1.3–1.1–1 are oversimple explanations of a complex argument. A detailed analysis should be as follows:

Proof. By the proof of Watson 109, p.14, (1), \( \sum_{n=0}^{\infty} a_n \cos^2(n\pi/2) \cdot J_n(z) \) converges uniformly on \( |z| \leq B \), where \( B \) is an arbitrary positive constant.

Let \( F(n,t) = 2O_n(t) - \frac{iO_{n+1}(t)}{n+1} - \frac{tO_{n-1}(t)}{n-1} + \frac{2n\sin(n\pi/2)}{n^2-1} \).

\( J_0(z)[O_0(t) - tO_1(t)] + J_1(z)[2O_1(t) - \frac{1}{2}tO_2(t) + \frac{1}{2}] + \sum_{n=2}^{\infty} J_n(z) F(n,t) \) converges uniformly in \( |z| \leq r < R \leq |t| \) [Watson 109, p.272, 1.2–1.1–1].

Assume \( \exists n \geq 2 : F(n,t) \neq 0 \).

Let \( n_0 = \min\{n \geq 2 : F(n,t) \neq 0\} \).

(\( \sum_{n=0}^{\infty} a_n e^{\pi m+n} \) converges uniformly on \( |z| \leq 1 \), where \( m \in \mathbb{N} \)) \( \Rightarrow (\sum_{n=0}^{\infty} a_n z^n \) converges uniformly on \( |z| \leq 1 \). Similarly, \( F(n_0,t_0) + \sum_{n=n_0+1}^{\infty} [J_n(z)/J_n(t)] F(n,t_0) \) converges to 0 uniformly on a small neighborhood of \( z = 0 \), where \( t_0 \) satisfies \( F(n_0,t_0) \neq 0 \).

Since \( (|z| \to 0) \Rightarrow [J_n(z)] \to \frac{1}{n^2}(\frac{n}{\pi})^n \) [Watson 109, p.40, (8)], we reach a contradiction. □

Example 6.44. (Determine \( \arg(1-t) \) on a contour around the branch point \( t = 1 \))

We need a method rather than correct results. Any step coming from guess may lead to the desired result this time; it may not next time. For example, if the choice \( \arg(-1) = \pi \) can lead to the desired result, we want to know why we cannot choose \( \arg(-1) = -\pi \). Thus, if one provides correct results without a method, one may still make mistakes sometimes. Ten correct examples are not as good as one correct method. Only when a correct method is provided may we check if results are correct. When encountering a situation where a confusion may easily occur, we should deliberately clarify the confusion rather avoid discussing it.

Example 1. After the circuit \((1+)\), \( \arg(1-t) = 2\pi \) [Watson–Whittaker 108, p.257, 1.1–1.2]].

Proof. \( t = 1 + \delta \exp(is) \), where \( \delta > 0, -\pi \leq s \leq \pi \).

\( 1-t = -\delta \exp(is) = \delta \exp[i(s+s_0)] \).

Before the circuit \((1+)\), \( s = -\pi \) and \( s+s_0 = \arg(1-t) = 0 \) [Watson–Whittaker 108, p.257, 1.1]]. Hence \( s_0 = \pi \).

After the circuit \((1+)\), \( s = \pi \). So \( \arg(1-t) = s+s_0 = 2\pi \).

Example 2. \( H_{v}^{(1)}(z) = \frac{2(z/2)^v}{\Gamma(v(1/2))} \int_{1+i\omega}^{1-i\omega} e^{iz} (1-t)^{v-1/2} dt \) [Watson 109, p.170, 1.13]]

Proof. \( H_{v}^{(1)}(z) = \frac{\Gamma(1/2-v)(z/2)^v}{\Gamma(1/2)} \int_{1+i\omega}^{1-i\omega} e^{iz} (t^2-1)^{v-1/2} dt \) [Watson 109, p.166, (4)]].

\( \int_{1+i\omega}^{1-i\omega} e^{iz} (t^2-1)^{v-1/2} dt \)

\( = (1-e^{2\pi i(v-1/2)}) \int_{1+i\omega}^{1-i\omega} e^{iz} (t^2-1)^{v-1/2} dt \).
Remark. Every formula for $H_{1}^{(1)}(z), J_{\nu}(z)$ and $Y_{\nu}(z)$ given in Watson [109] p.170, l.−18−l.−1 should have been added the factor 2 on the right-hand side of its equality. The step given in Guo–Wang [46] p.371, 1.8 gives the result, but fails to provide a method of getting the answer. According to the way that the solution is approached, very likely the reasoning contains guesses, and thus may be incorrect.

Example 6.45. (Binomial series)
The classical view emphasizes the choice of principal value and the consistency with previous results: The principal value of the power of a binomial given in Hobson [52] p.325, 1.1 draws my attention. I find its principal value of the power of a binomial comes from the principal value of the logarithm [Hobson [52], p.269, 1.−1]. The principal value of $z^{\alpha}$ is defined by $z^{\alpha} = e^{\alpha \log z}$, where $\log$ is the principal branch of the logarithmic function. Thus, the number of values of $z^{\alpha}$ is finite if $\alpha$ is a rational number and is countably infinite if $\alpha$ is an irrational number. Hobson [52] p.269, l.−3–l.−1 shows that the binomial series $f(m)$ given in Hobson [52] p.268, 1−9 converges to the principal value of $(1+z)^{m}$ when $m$ is a positive rational number $p/q$. When $m$ is a positive irrational number, $f(m) = Lf(mv)$ [Hobson [52] p.270, 1.7] = the limit of the principal value of $(1+z)^{m}$

= the principal value of $(1+z)^{m}$ [by continuity or Rudin [88], p.225, Theorem 10.18].

The modern view emphasizes whether $\sum_{k=0}^{\infty} (\alpha\choose k) z^{k}$ is convergent and whether the cases considered are inclusive: By the ratio test [Rudin [86] pp. 57–58, Theorem 3.34], we have

Theorem 1. Suppose $\alpha$ is not a non-negative integer. Then the radius of convergence for $\sum_{k=0}^{\infty} (\alpha\choose k) z^{k}$ is 1.

Theorem 2. Let $C = \{ z \in C : |z| = 1 \}$. Then

1. $\sum_{k=0}^{\infty} (\alpha\choose k) z^{k}$ converges at all points on $C$ if $\Re \alpha > 0$.
2. $\sum_{k=0}^{\infty} (\alpha\choose k) z^{k}$ diverges at all points on $C$ if $\Re \alpha < 0$.
3. $\sum_{k=0}^{\infty} (\alpha\choose k) z^{k}$ diverges at $z = 1$ and converges at all other points on $C$.

Proof: $(-1)^{k}\cdot (\alpha\choose k) = (-\alpha+k-1)$. By Guo–Wang [46] p.97, (3)], we have

$$(\alpha\choose k) = \frac{(-1)^{k}}{1 + o(1)}(\frac{1}{k!})^{m}$$. Consequently,

$$(\alpha\choose k) \leq \frac{1}{k^{m+\alpha}}(1+o(1)) \leq \frac{M}{k^{m+\alpha}} \quad (*)$).

(i) follows from $(*)$, by comparison with the p-series $\sum_{k=1}^{\infty} k^{-p}$, where $p = 1 + \Re \alpha$.

(ii). $|\alpha\choose k| \leq \frac{1}{k^{m+\alpha}} \quad (**)$. If we let $z = -1$ and replace $\alpha$ with $\alpha - 1$ in $(**)$, we have

$\sum_{k=0}^{\infty} (-1)^{k} = \frac{(-\alpha-1)(-1)^{n+1}}{n}$.

Example 6.46. (Listing examples cannot be considered a proof)

Listing examples cannot be considered a proof just like a tangled ball of yarn cannot be called a piece of cloth. A professional proof must give the direction of thoughts and the key idea. We should not avoid discussing the part difficult to discribe. On the contrary, we should work harder to give it a clear explanation.

Example 1. Hobson [52] p.51, l.−21−l.−6] fails to provide a proof except stating the two formulas and providing a few examples. A professional proof should be as follows:

80
However, \(A_{n/2} = \sum \cos B\), where \(B\) is the sum of \(k\) positive angles and \(k\) negative angles.

2\(\cos B\sin A_{n+1} = \cos(A_{n+1} + B) + \sin(A_{n+1} - B)\).

\(A_{n+1} + B\) keeps the \(k\) negative angles in \(B\), but cannot change the \(k\) positive angles in \(B\) into negative angles. However, \(A_{n+1} - B\) can change the \(k\) positive angles in \(B\) into negative angles. Consequently, there is no coefficient \(1/2\) in front of \(D'_{(n+2)/2}\).

II. How \(D'_{(n+2)/2}\) becomes \(\frac{1}{2}C''_{(n+2)/2}\).

Proof. \(D'_{(n+2)/2} = \sum \sin B\), where \(B\) is the sum of \(k + 1\) positive angles and \(k\) negative angles.

2\(\sin B\sin A_{n+1} = \cos(B - A_{n+1}) - \cos(B + A_{n+1})\).

The \(k + 1\) negative angles in \(B - A_{n+1}\) contains the \(k\) negative angles in \(B\) and \((-A_{n+1})\), but not the positive angles in \(B\). Therefore, we must add the coefficient \(1/2\) in front of \(C''_{(n+2)/2}\).

II. How \(D'_{(n+2)/2}\) becomes \(\frac{1}{2}C''_{(n+2)/2}\).

Proof. \(D'_{(n+2)/2} = \sum \sin B\), where \(B\) is the sum of \(k + 1\) positive angles and \(k\) negative angles.

2\(\sin B\sin A_{n+1} = \cos(B - A_{n+1}) - \cos(B + A_{n+1})\).

The \(k + 1\) negative angles in \(B - A_{n+1}\) contains the \(k\) negative angles in \(B\) and \((-A_{n+1})\), but not the positive angles in \(B\). Therefore, we must add the coefficient \(1/2\) in front of \(C''_{(n+2)/2}\).

A finite series must have the first term, the last term, and the general term. An infinite series must have the first term and the general term. Thus, the two formulas given in Hobson [52] p.107, l.2 & (7)] are not correctly presented. To figure out the general term of a series from its first few terms is an example of inductive reasoning or a conjecture, but should not be considered a proof. For a binomial coefficient, we should use its compact symbol \(\binom{n}{k}\) rather than its awkward factorial form \(\frac{n!}{k!(n-k)!}\) unless for the purpose of computation. The formulas given in Hobson [52] p.106, l.8–l.1–l.1] look messy due to the abuse of notation. Suppose \(n\) is even. Hobson [52] p.107, (7)] expresses \(\cos n\theta\) as a finite series in ascending powers of sine without the highest power term. Hobson [52] p.105, (3)] expresses \((-1)^{n/2} \cos n\theta\) as a finite series in descending powers of sine without the lowest power term. Hobson [52] p.107, l.12] claims that Hobson [52] p.107, (7)] is Hobson [52] p.105, (3), written in reverse order. How is it possible to compare two things when one of them is unknown? Example 2.

(a). A neat presentation of Hobson [52] p.107, l.2–l.3] should be as follows:

\[\binom{p+q}{s} = s!\sum_{k=0}^{s} \binom{s}{q} \binom{s-k}{p} \quad [\text{Zhu–Vandermonde’s identity}]
\]

(b). Hobson [52] p.107, (7)] should have been corrected as follows:

When \(n\) is even,

\[\cos n\theta = \sum_{k=0}^{n/2} (-1)^{k} n^2(n^2-2^2-2(n^2-1)^2/k!) \sin^{2k} \theta.
\]

(c). A neat presentation of Hobson [52] p.106, l.9–l.1–l.1] should be as follows:

When \(n\) is even,

\[\cos n\theta = \sum_{0 \leq k \leq n/2} (-1)^{k} n^2 \binom{n}{2k} (1 - \sin^2 \theta)^{n/2-k} \sin^{2k} \theta
\]

\[= \sum_{k=0}^{n/2} \binom{n}{2k} \sum_{k=0}^{n/2} \binom{n}{2k} \binom{n/2-k}{k} \sin^{2k} \theta
\]

Only the proof of the last equality requires the use of the factorial form of binomial coefficients for computation.

(d). Suppose \(n\) is even. Hobson [52] p.107, (7)] should have been corrected as
“\( \cos n\theta = \sum_{s=0}^{n/2} (-1)^s \sin^{2s} \theta \left( \frac{n!}{(n-2s)!(2s)!} \right) \)”,

Hobson [52, p.105, (3)] should have been corrected as

“\((-1)^{n/2} \cos n\theta = 2^{n-1} \sin^n \theta + \sum_{r=0}^{n/2} (-1)^r \sin^{n-2r} \theta \left[ \frac{2^r}{(n-r-1)!} \right] \)”,

By replacing \( r \) with \( n/2 - r \) in the general term of Hobson [52, p.105, (3)], we will obtain the general term of Hobson [52, p.107, (7)].

(c). Hobson [52, §78 & §79] expresses \( \cos n\theta \) and \( \sin n\theta \) as descending power series of sine. Their combinatorial proofs are tedious and annoying. If we want to express them in ascending power series of sine, all have to do is list all the terms of the descending power series and then reverse the order. However, Hobson [52, §80–§83] fails to do this simple way by repeating the same kind of tedious and annoying combinatorial proofs. Mathematics is not for killing time. We have more important things to do.

(f). (Clarification of a point of confusion)

\( \cos n\theta = \sum_{s=0}^{n} (-1)^s \sin^{2s} \theta \left( \frac{n!}{(n-2s)!(2s)!} \right) \), where \( n \) is a positive integer [Hobson [52, p.274, 1.5–1.6]].

**Proof.**

First, let us clarify the relationships among Hobson [52, §78, (1), §79, (3), §80, (7), §214, (5)]. Hobson [52, §214, (5)] and Hobson [52, §80, (7)] are the same. Hobson [52, §80, (7)] is Hobson [52, §79, (3)], written in reverse order. Hobson [52, §79, (3)] is a special case of Hobson [52, §78, (1)] when \( n \) is even.

If we replace \( \cos \theta \) with \( 1 - \sin^2 \theta \)\(^{1/2} \) in Hobson [52, §78, (1)], after expansion and rearrangement we will obtain Hobson [52, §80, (7)] when \( n \) is even. However, the symbol \( n \) in Hobson [52, §78, (1)] can represent an odd or even integer and \( \{ \sin^k \theta | n \in \{0\} \cup \mathbb{N} \} \) are linearly independent, so Hobson [52, §80, (7)] is true when \( n \) is odd.

Corollary. \( \frac{d^{2k+1}}{d\theta^{2k+1}} \left[ \cos n\theta - 1 - \sum_{s=1}^{k} (-1)^s \cos^{2s} \theta \left( \frac{n!}{(n-2s)!(2s)!} \right) \right] \big|_{\theta=0} = (-1)^{k+1} \left[ n^2 (n^2 - 2^2) \cdots (n^2 - [k^2]) \right] \).

**Example 6.47.** (The proof of a theorem is hidden in the application whose proof requires the use of the theorem)

Example. (Stirling’s theorem)

\( \ln \Gamma(s+\lambda) = (s+\lambda - \frac{1}{2}) \ln s - s + \frac{1}{2} \ln (2\pi) + O(s^{-1}) \) [Guo–Wang [46, p.155, (6)].]

**Proof.**

By Watson–Whittaker [108, §13-6], we obtain

\( \ln \Gamma(s+\lambda) = (s+\lambda - \frac{1}{2}) \ln s - s + \frac{1}{2} \ln (2\pi) + O(s^{-1+\eta}) \) (*),

where \( \eta \) is an arbitrary small positive number. By the existence and uniqueness of the Laurent series, we have Guo–Wang [46, p.155, (6)].

**Remark 1.** I had been unable to prove Guo–Wang [46, p.155, (6)] until I attempted to prove the formula given in Watson [109, p.225, 1.12–1.11]. All I could do was prove (*) because for most applications \( \ln \Gamma(s+\lambda) = (s+\lambda - \frac{1}{2}) \ln s - s + \frac{1}{2} \ln (2\pi) + O(1) \) (***) is sufficient. For example, we can use (**) to prove Guo–Wang [46, p.155, (7)]. However, the proof of the formula given in Watson [109, p.225, 1.12–1.11] requires the use of Guo–Wang [46, p.155, (6)]. In this case, (*) is not good enough for the high accuracy. I am able to complete the proof of Guo–Wang [46, p.155, (6)] because the second factor of the right-hand side of the formula given in Watson [109, p.225, 1.12–1.11] is a Laurent series.

**Remark 2.** “Eq. (5) of Sec. 3.2” given in Guo–Wang [46, p.155, 1.1] should have been corrected as “Eq. (5) of Sec. 3.21”.

**Example 6.48.** (Finding the inverse function of a given analytic function with the Fourier series method)

The inverse function theorem provides the existence of inverse function, but fails to provide an algorithm.

Remark 1. Without reading the section “A Fourier Sine Series Expansion and Resulting Bessel Function Representation for the Coefficients” in http://www.murison.alpheratz.net/Maple/KeplerSolve/KeplerSolve.pdf, I would still be puzzling over how Bessel could have obtained the integral given in Watson [109] p.19, l.1–6.

Remark 2. \( B_n = -2(\varepsilon/n)J'_n(n\varepsilon) \) [Watson [109] p.6, l.1–17].

\[ \begin{align*}
\text{Proof.} & \quad \text{Let } -\varepsilon \cos E = \sum_{n=0}^{\infty} B_n \cos(nM). \\
& \quad B_0 = -\frac{1}{2} \int_0^\pi \varepsilon \cos EdM \\
& \quad = -\frac{1}{2\pi} \int_0^\pi \varepsilon \cos E(dE - \varepsilon \cos EdE) \quad \text{(because } M = E - \varepsilon \sin E) \\
& \quad = \varepsilon^2/2.
\end{align*} \]

Assume \( n \neq 0 \).
\[ \begin{align*}
B_n &= -\frac{2}{\pi} \int_0^\pi \varepsilon \cos E \cos nM dM \\
&\quad = -\frac{2}{\pi} \int_0^\pi \varepsilon \sin nM \sin EdE \quad \text{(integration by parts)} \\
&\quad = -\frac{2\varepsilon}{\pi} \int_0^\pi \sin(E - \varepsilon \sin E) \sin EdE \\
&\quad = -2(\varepsilon/n)J'_n(n\varepsilon) \quad \text{[Watson [109] p.19, l.1–6].}
\end{align*} \]


\textbf{Example 6.49.} (Statements of a certain type have the same proof pattern)

Let \( L \) be a number and \( (a_n) = (a_0,a_1,\cdots) \) be a number sequence. Then the following statements belong to the same type and have the same proof pattern.

\[ \{ (a_n) | \lim_{n \to \infty} A_n = L \} \subseteq \{ (a_n) | \lim_{n \to \infty} [A_0 + A_1 + \cdots + A_{n-1}]/n = L \} \]

\[ \supseteq \{ (a_n) | \lim_{n \to \infty} [nA_0 + (n-1)A_1 + \cdots + A_{n-1}]/[n(n+1)/2] = L \} \quad \text{[Bromwich [16] p.132, l.1–1].} \]

\[ \begin{align*}
\text{Proof.} & \quad \text{I. } \lim_{n \to \infty} A_n = L \Rightarrow \lim_{n \to \infty} [A_0 + A_1 + \cdots + A_{n-1}]/n = L. \\
\text{Proof.} & \quad \frac{A_0+A_1+\cdots+A_{n-1}}{n} - L = \frac{(A_0-L)+(A_1-L)+\cdots+(A_{n-1}-L)}{n}. \quad \blacksquare
\end{align*} \]

II. \( \lim_{n \to \infty} [A_0 + A_1 + \cdots + A_{n-1}]/n = L \Rightarrow \lim_{n \to \infty} [nA_0 + (n-1)A_1 + \cdots + A_{n-1}]/[n(n+1)/2] = L. \)

\[ \begin{align*}
\text{Proof.} & \quad [nA_0 + (n-1)A_1 + \cdots + A_{n-1}]/[n(n+1)/2] - L \\
&\quad = \frac{A_0+A_1+\cdots+(A_0+A_1+\cdots+A_{N-1})-[N_0(N_0-1)/2]L}{n(n+1)/2} + \frac{N_0}{\pi} \frac{(A_0+A_1+\cdots+A_{N-1})-L}{(n+1)/2} \quad \blacksquare
\end{align*} \]

III. Bromwich [16] p.133, Ex.4, (iv)] shows that \( \{ (a_n) | \lim_{n \to \infty} A_n = L \} \setminus \{ (a_n) | \lim_{n \to \infty} A_n = L \} \neq \emptyset. \)
Remark. \lim_{n \to \infty} \sum_{k=1}^{\infty} a_k = L \Rightarrow \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \frac{1}{n} \sum_{k=1}^{n} a_k = L.

Proof. By Watson–Whittaker [108, p.33, l.3–l.4],

\sum_{n=1}^{\infty} 1 = \frac{1}{1-q^{2n-1}e^{-2\pi\tau}} \sum_{n=1}^{\infty} \frac{2q^{2n-1}e^{2\pi\tau}}{1+q^{2n-1}e^{2\pi\tau}} [\text{Watson–Whittaker [108 p.471, 1.6]}].

Remark. Sometimes, only after studying advanced mathematics may we understand how we should properly deal with elementary mathematics. In order to study infinite products of analytic functions, we must master the concept of uniform convergence. Thus, it is important to see how the Taylor series and the L’Hôpital rule affect convergence. Among proofs for the case of point convergence, we should select the ones applicable to the case of uniform convergence. Watson–Whittaker [108, p.33, l.1–l.7] shows that the absolute convergence of \sum a_n is equivalent to that of \sum a_n using the Taylor series. The proof is applicable to the case of uniform convergence. The section “Convergence criteria” of https://en.wikipedia.org/wiki/Infinite_product proves that the convergence of \sum \log(1 + a_n) is equivalent to that of \sum a_n using the L’Hôpital rule. The proof is not applicable to the case of uniform convergence because \lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)} refers to a single point z = a.

Example 6.50. (The Taylor series vs. the L’Hôpital rule in terms of convergence)

\vartheta_3(z) = \vartheta_3(z) \sum_{n=1}^{\infty} \frac{2q^{2n-1}e^{2\pi\tau}}{1+q^{2n-1}e^{2\pi\tau}} \sum_{n=1}^{\infty} \frac{2q^{2n-1}e^{-2\pi\tau}}{1+q^{2n-1}e^{-2\pi\tau}} [\text{Watson–Whittaker [108 p.469, 1.2]}].

Proof. By Watson–Whittaker [108, p.33, l.3–l.4],

\sum_{n=1}^{\infty} \log(1 + 2q^{2n-1}e^{2\pi\tau}) \log(1 + 2q^{2n-1}e^{-2\pi\tau}) | converges uniformly on compact subsets of \mathbb{C} \{\text{the zeros of } \vartheta_3(z)\}.

1 + 2q^{2n-1}e^{2\pi\tau} + 2q^{2n-2} = (1 + q^{2n-1}e^{2\pi\tau})(1 + q^{2n-1}e^{-2\pi\tau}),

The desired result follows from Rudin [88, p.230, Theorem 10.28].

Remark. In the last step, one should not expand the expressions on the numerator of the previous step. Cancel the common factor of the numerator and the denominator first. This may avoid a lot of unnecessary computations. If one were to expand the expressions in the numerator of the first step, the resulting \sum_{n=1}^{\infty} a_n would be \frac{64z^2 + 368z^5 - 1032z^7 + 273z^9 - 371z^{11} - 25z^{13}}{128(1 - z^5)^2(z^4 + 3z^6 + 4z^8 + 2z^{10} + 3z^{12} + 2z^{14} + z^{16})}. The complicated expression would make it more difficult to identify it with the value given in Watson [109, p.226, l.8].
Example 6.52. (The motive of creation and process of evolution for the method of steepest descents)

The interpretation of the method of steepest descents from the viewpoint of physics is the simplest and most direct [The principle of stationary phase: Watson [109] p.230, 1.15–1.18]. The interpretation from the viewpoint of mathematics is given by Guo–Wang [46] §7.11. Watson [109] p.238, Fig. 16] [Case $x/v < 1$] shows that the integral contour given in Watson [109] p.176, (3)] can be replaced with the steepest descent without changing the lower limit and the upper limit of the integral; Watson [109] p.239, Fig. 17] [Case $x/v > 1$] shows that each integral contour in Watson [109] p.178, (2) & (3)] can be replaced with the steepest descent without changing the lower limit and the upper limit of the integral; Watson [109] p.240, Fig. 18] [Case $x/v = 1$] shows that each integral contour in Watson [109] p.176, (3); p.178, (2) & (3)] can be replaced with the steepest descent without changing the lower limit and the upper limit of the integral. The arrows given in Watson [109] p.238, Fig. 16; p.239, Fig. 17; p.240, Fig. 18] can be explained by Guo–Wang [46] p.383, 1.3–1.4; l.8–1.9]. The τ’s given in Watson [109] p.238, 1.4; p.239, 1.11] come from Guo–Wang [46] p.384, l.6. The following path shows the process of evolution for the method of steepest descent: Abel’s test $f(x,y)$ decreases as $x$ increases, the upper limit of each integral is $\infty$; Bromwich [16, p.434, l.4–p.435, l.3]]

$I. \lim_{n \to \infty} \text{ }$\[f(x,n) \text{ decreases as } x \text{ increases, } f(x,n) \to g(x) \text{ uniformly in any fixed finite interval, the upper limit of integral tends to } \infty \text{ as } n \to \infty; \\
\text{Bromwich [16, p.443, 1.6–1.11]}

$\rightarrow [f(x,n) \text{ decreases as } x \text{ increases, } f(x,n) \to g(x) \text{ uniformly in any fixed finite interval, the upper limit of integral tends to } \infty \text{ as } n \to \infty; \\
\text{Bromwich [16, p.443, 1.6–1.11]}

$\rightarrow$ The extension to the case when $f$ has a limit number of maxima and minima [The condition that $f$ is positive and never increasing is removed; Bromwich [16, p.444, l.6–1.4–1.41]]

$\rightarrow [F \text{ is a function of bounded variation, the upper limit of integral tends to } \infty \text{ as } n \to \infty; \\
\text{Watson [109, p.230, 1.12–1.9]]}

$\rightarrow$ Watson’s lemma [Watson [109, p.236, 1.11–1.19]; Guo–Wang [46, p.34, Watson’s lemma]]


Remark 1. $\int_0^\infty \sin t \cdot t^{n-1} dt = \Gamma(n) \sin(\frac{\pi}{2}n\pi)$, where $0 < n < 1$ [Bromwich [16, p.447, 1.10; p.474, 1.11]; Watson [109, p.230, 1.10–1.10]].

Proof. $I. \lambda = 0, \xi > 0, \eta = \xi + i\eta, \text{ and } U = \int e^{-\lambda t} t^{n-1} dt. \text{ Then } U(x) = \Gamma(n)/\xi^n.$

$\partial U \partial \eta = -\frac{1}{2} e^{-\lambda} U \partial U \partial \eta = -\frac{i}{2} e^{-\lambda} U.$

When $\eta = 0, \Gamma(n)/\xi^n.$

$\Gamma(n)/\xi^n.$ satisfies the above system of partial differential equations and Cauchy data.

By the Cauchy–Kowalevski theorem [John [57, p.74, 1.4–1.5]], $U(x) = \Gamma(n)/\xi^n$ locally along the positive real axis.

By analytic continuation, $U(x) = \Gamma(n)/\xi^n$ in the half-plane $\xi > 0.$

II. $\int_0^\infty e^{-\lambda t} t^{n-1} dt = \lim_{\epsilon \to 0^+} \int_0^\infty e^{-\lambda t} t^{n-1} dt [\text{Bromwich [16, p.436, Ex. 2]}]$ $\Gamma(n)$ [by I]

$= \lim_{\epsilon \to 0^+} \frac{\Gamma(n)}{e^{\lambda t}}$ [by I]

$= \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}\right) \Gamma(n).$

Remark 2. Guo–Wang [46, p.382, 1.16–p.384, 1.3] provides a local view of the method of steepest descent, while Born–Wolf [13, p.885, l.10–p.886, l.–1.16] provides a global view. All the three sections in Born–Wolf [13, Appendix III] are based on the same idea. Why do we change variables? How do we change the variable to the one suitable for approximating the integral? The answers are clear in $t \to \rho$ [Guo–Wang [46, p.382, l.–5; p.383, 9]; l.–12] and $(x) \to (\xi, \eta)$ in order to eliminate $\cdots$ in Born–Wolf [13, p.890,
Example 6.53. (With vs without guess and check)

Proofs are used to check the truth of a statement and are not necessarily helpful to understand its meaning. For example, we can use the mathematical induction to prove $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, but do not know how we get this formula. The proof is independent of the theme of this formula in the same way as a quality control inspector checks only the packaging of product. This is a proof with guess and check; its analysis for the formula is shallow. We make the conclusion without enough confidence beforehand, and have to check afterwards; the guess and poor explanation lowers the quality of theory. Therefore, ideal and mature mathematical theories should gradually eliminate the guesswork in it. For example, the first answer in https://math.stackexchange.com/questions/183316/how-to-get-to-the-formula-for-the-sum-of-squares-of-first-n-numbers is a proof without guess and check. Its features: having a specific viewpoint; starting with a careful plan to get the answer; all the operations being in control beforehand. The second answer in https://math.stackexchange.com/questions/183316/how-to-get-to-the-formula-for-the-sum-of-squares has some guesswork in the beginning. Hobson [52, p.48, (28) & (29)] are guessed from the case $n = 1, 2, 3$, and then proved by the mathematical induction, while Bromwich [16, §66 – §68] are without guess and check; their discussion starts with a fixed method, the rest of discussion is just the execution of calculations.

Example 6.54. ( Infinite integrals)

Mathematics tends to be more effective in development [Bromwich [16, p.429, l.2–l.10]] the same way living beings tends to be more intelligent in evolution. The theory of infinite integrals emphasizes effective tests and evaluation.

(Abel’s lemma) There are two proofs for Abel’s lemma given in Bromwich [16, p.426, l.5–l.11]. Case when $f$ is differentiable: the proof given in Bromwich [16, p.426, l.12–l.19] uses integration by parts. Case when $f$ is non-differentiable: the proof given in Bromwich [16, p.427, l.1–l.15] uses summation by parts. The two proofs are based on the same idea.

(Absolute convergence vs convergence for alternating series [Bromwich [16, p.428, l.13–l.17] & Watson [109, p.230, l.10]] are without guess and check; their discussion starts with a fixed method, the rest of discussion is just the execution of calculations.

(Tests of convergence) The convergence of absolute convergence vs convergence for alternating series [Bromwich [16, p.428, l.13–l.17] & Watson [109, p.230, l.10]] are without guess and check; their discussion starts with a fixed method, the rest of discussion is just the execution of calculations.

(Dirichlet’s test) The statement of Dirichlet’s test is given in Bromwich [16, p.429, l.2–l.10]. The proof is given in Buck [17, p.218, l.12–l.17]. Although the assumption of Buck [17, p.218, Theorem 17] is more restrictive, we may easily remove the restrictions [continuity of $g$; $\int_{c}^{\infty} |g'| < \infty$] by considering $\int_{c}^{\infty} |g|$. The proof given in Bromwich [16, p.430, l.1–l.13] and that given in Watson–Whittaker [108, p.72, l.4–l.7] are good for only the case when $f$ is decreasing. If $f$ is increasing, we must consider $-f$.

(Tests of uniform convergence: the method of change of variable) Watson–Whittaker [108, p.72, l.1–l.11–p.73, l.2] gives a typical example. The proof of uniform convergence for each of the three integrals given in
Bromwich [16, p.436, 1.6] is similar.

(Lebesgue’s dominated convergence theorem) The proof of the theorem given in Bromwich [16, p.438, 1.16–1.12] is similar to but simpler than that of Rudin [88, pp.246–247, Exercise 16]. The following statements can be considered corollaries of Lebesgue’s dominated convergence theorem (LDCT) [Rudin [88, p.27, Theorem 1.34]]. For each statement, I indicate only the place where LDCT is used.

By Bromwich [16, p.441, Ex. 2],
\[ \sum_{a} = \int_{-\infty}^{\infty} \text{integral tends to} \]
Bromwich [16, p.441, l.1] says that \( \lim_{t \to 0} \int_{a}^{b} f(x) \, dx \) is uniformly convergent for \( f \) is analytic at \( x = 0 \).

Weierstrass’ test \([ \text{The process of evolution for Abel’s test for uniform convergence vs that for Weierstrass’ test} \) [Bromwich [16, Art. 172, (1)]] can be proved by LDCT. In my opinion, it is more direct to say that \( \lim_{t \to 0} \int_{a}^{b} f(x) \, dx \) is uniformly convergent for an integrable function.

The extension to the case when \( f \) has alimit number of maxima and minima [The condition that \( f \) is positive and never increasing is removed; Bromwich [16, p.444, l.6–l.11]]

\( \rightarrow [F \text{ is a function of bounded variation, the upper limit of integral tends to } \infty \text{ as } n \to \infty; \text{Watson} [109, p.230, 1.12–1.9]\]
\( \rightarrow \) Watson’s lemma [Watson [109, p.236, 1.11–1.19]; Guo–Wang [46, p.34, Watson’s lemma]]
\( \rightarrow \) the method of steepest descent [Guo–Wang [46, p.384, 1.11–1.13]]

Remark 1. \( J = \int_{0}^{\infty} x^{2} e^{-xy} \left[ \sum_{a} (1 + ax) + a_{2} x \right] e^{-ax} \, dx \). Then \( J \) is uniformly convergent for \( y \geq 0 \).

Note that \( \sum_{a} (1 + ax) + a_{2} x \) is analytic at \( x = 0 \).

\( J(y) = - \int_{0}^{\infty} x^{2} e^{-xy} \left[ \sum_{a} (1 + ax) + a_{2} x \right] e^{-ax} \, dx \). This integral converges uniformly so long as \( y \geq l > 0 \).

By Bromwich [16, p.441, Ex. 2], \( J(y) = \int_{0}^{\infty} \log(a + y) - \frac{1}{a + y} \, dy \).

\[ \lim_{y \to \infty} J = 0. \]

\( \int_{0}^{\infty} x^{2} e^{-xy} \, dx = \frac{\sin(xy)}{1 + x^{2}} \), converges uniformly in \( y \geq l > 0 \).

Remark 2. \( J \) remains finite as \( y \) tends to \( \infty \) [Bromwich [16, p.442, 1.4]].

Proof. By the method of change of variable, \( J'' = - \int_{0}^{\infty} x \sin(xy) \, dx \) converges uniformly in \( y \geq l > 0 \).
\[ \frac{d^2 J}{dx^2} - J = -\frac{\pi}{2} \] [Bromwich [16] p.442, 1.2].

Remark. It would be difficult to prove the above statement if we were to consider \( J \) alone because \( \lim_{x \to 0} \frac{\sin(xy)}{x} = y \).

**Example 6.55.** (A science book author should not use definitions to stop readers’ questions)

For any science book, a reader should not accept a definition as a command about whose origin one should not question although it does not require a proof. An author should not give a definition without providing a reason. The definition given in Cohen-Tannoudji–Diu–Laloe [23] vol. 2, p.1476, (48)] fails to clearly explain from where it comes. In contrast, the formula given in Born–Wolf [13] p.894, 1.−4] explains why we define the derivative of \( \delta \) as in Born–Wolf [13] p.895, (13). This formula is established using the method of integration by parts. In fact, this method is the key to building distribution theory [Rudin [87, p.136, (1) & (3)]]. The problem \( f(\infty) \delta(\infty, \mu) = 0 \) [Born–Wolf [13] p.894, (1.−4); p.895, l.1] may be solved by restricting the testing functions \( f \) to the domain \( \mathcal{D} \) [Rudin [87] p.136, 1.9]].

It is special and interesting that the introduction to \( \delta \)-function in the Fourier-transform form [Born–Wolf [13] p.896, (23)] begins with the Fourier integral theorem [Born–Wolf [13] p.895, (19); Rudin [87, p.170, Theorem 7.7 (a)]].

**Example 6.56.** (The right timing for correcting mistakes)

In physics, we study facts. Theories are nothing but tools to explain facts. When a theory fails to explain facts, it should be abandoned and eliminated. When we find a statement contradictory to facts, we should trace to the origin of mistake and rewrite the theory from there. For a system of identical particles, the formula given in Pathria–Beale [77] p.15, (21) is incorrect because (i) the entropy is not an extensive property of the system [Pathria–Beale [77] p.16, l.−1−p.17, l.1] and because (ii) Pathria–Beale [77] p.17, (4) contradicts Pathria–Beale [77] p.18, (4a)]. Pathria–Beale [77] p.15, (21) should have been corrected as Pathria–Beale [77] p.19, (1a)). This is because \( \Gamma \) [Pathria–Beale [77] p.14, (20)] should have been divided by \( N! \) when the particles in the system are identical. However, Pathria–Beale [77] p.18, l.1−l.−3] fails to follow this simple and direct approach. Instead, it makes a fuss about it by showing the consequences if we were to assume Pathria–Beale [77] p.15, (21)) is true and by providing a remedy to satisfy the requirement. It is deplorable that Pathria–Beale [77] p.18, l.−9−l.−3] still fails to point out why the remedy can work. Of course, an incorrect statement will lead to a lot of junks, but we are not interested in why they are junks. The important thing is to correct mistakes as soon as they occur. Perhaps the Gibbs paradox is valuable for books about the development history of statistical mechanics, but not for a textbook.

How do we choose the right timing for correcting mistakes? Shall we do it when we start to count the number of microstates [Pathria–Beale [77] p.10, l.8−l.9]? No. If we were to consider a system of identical particles too early, we would encounter the difficulty given in Pathria–Beale [77] p.11, l.−3−l.−1]. Shall we correct mistakes at the position of Pathria–Beale [77] p.19, l.3−l.4]. No. If we were to consider a system of identical particles and correct the mistake so late, then the validity of all the statements between Pathria–Beale [77] p.14, l.2] and Pathria–Beale [77] p.19, l.2] would become questionable. A textbook should not contain any incorrect statement because it is a reference book for quotation and application.

**Example 6.57.** (Maxwell made a contradiction compatible by changing \( \nabla \times H = J_f \) to \( \nabla \times H = J_f + J_d \))

The contradiction to be resolved: \( \nabla \times H = J_f \) [Wangness [106] p.348, (21-1)] leads to Wangness [106 p.348, (21-3)], which contradicts Wangness [106, p.15, (1-49)].

His analysis: Because Wangness [106 p.15, (1-49)] is a mathematical theorem, it must be true. Most likely, the problem arises from the incompleteness of the formula \( \nabla \times H = J_f \).
His remedy for compatibility: Consequently, Maxwell changes it to $\nabla \times H = J = f + J_d$ [Wangness [106] p.348, (21-4)]. Then he uses the two formulas [Wangness [106] p.152, (10-41); p.207, (12.19)] to obtain $\nabla \cdot (\nabla \times H) = -\frac{\partial}{\partial t} \nabla \cdot D + \nabla \cdot J_d$ [Wangness [106] p.348, 1–1]. The derivation of these two formulas is impeccable and we cannot do nothing about these universal principles. Then we have $J_d = \frac{\partial D}{\partial t}$ [Wangness [106, p.349, l.1–5]]. Therefore, the formula $\nabla \times H = J = f$ should be corrected as $\nabla \times H = J = f + \frac{\partial D}{\partial t}$ [Wangness [106, p.349, (21-7)].

How the correction of the formula affects the results whose validity depends on the formula: Wangness [106, p.349, l.13–l.15].

Other evidence of the existence of displacement current: case $\rho > a$: if the displacement current did not exist, we would get a contradiction to the boundary condition for tangential components of $H$ [Wangness [106, p.352, l.4–l.13]]; $\rho < a$: Wangness [106, p.352, (21.17)] agrees with the boundary condition for tangential components of $H$ because we include the displacement current as a source of $H$ [Wangness [106, p.352, l.14–p.353, l.7]].

Example 6.58. (Faraday made a contradiction compatible by changing $\nabla \times E = 0$ [static] to $\nabla \times E = -\frac{\partial B}{\partial t}$ [nonstatic])

Proof. Assume that $\nabla \times E = 0$ is applicable to the nonstatic case. Wangness [106, p.264, (17-3)] [by experiments]

$\Rightarrow$ Wangness [106, p.266, (17-7)] [by Wangness [106, p.266, (17-6)]]

$\Rightarrow$ Wangness [106, p.266, (17-8)] [by Wangness [106, p.251, (16-6)].

Case of stationary media: Wangness [106, p.266, 1.–5–p.267, 1.8];

Case of moving media: Wangness [106, p.269, 1.–6–p.272, 1.10]). Thus, We have a contradiction.

Remark 1. How the correction of the formula affects the results whose validity depends on the formula:

I. Stationary media: Wangness [106, p.267, 1.9–l.9];

II. Moving media: Wangness [106, p.272, 1.10–1.13];

locatize the source of $\delta_m^e$ and interpret the origin of induced current: Wangness [106, p.272, 1.6–1.1] from start to the equilibrium state described by Wangness [106, p.273, (17-34)];

Wangness [106, p.274, 1.3–l.11];

locate the portions that contribute to the induced emf: Wangness [106, p.276, 1.10–1.14];

homopolar generator: Wangness [106, p.276, 1.4–p.277, 1.4; p.277, 1.6–1.8].

Remark 2. Example 5.5.

Example 6.59. (How we should properly treat Ampère’s law)

I. The situation: magnetostatics, idealized circuits [Wangness [106, p.217, l.23–l.2]].

II. Our strategy: Like Coulomb’s law discusses charges, Ampère’s law discusses current elements [Wangness [106, p.217, l.2–p.218, 1.4]]; see Wangness [106, p.219, (13-6)]. This concept is used to provide a method of building a formula that will match experimental results [Wangness [106, p.218, 1.5–1.8]]. Applying Wangness [106, p.218, (13-1)] to Wangness [106, p.220, Figure 13-2] may quickly result in the desired direction $\hat{\rho}$ [Wangness [106, p.221, (13-10)] and a scalar double integral [Wangness [106, p.221, (13-11)].

If we use Wangness [106, p.219, (13-6)] instead, we will not have these advantages. As for other aspects of comparison between Wangness [106, p.218, (13-1)] and Wangness [106, p.219, (13-6)], see Wangness [106, §13-3].

II. The value of Ampère’s law: This law is not a mathematical theorem derived from axioms. Instead, it is a method which describes a natural phenomenon mathematically and which provides an algorithm for calculating the magnetic force.
Example 6.60. (A more delicate and effective method provides more information)

Based on Wngsness [106, p.52, Figure 3-1; p.74, Figure 5-5], we can derive Wngsness [106, p.75, (5-28)]. However, this method gives no information about $\rho_0$ for which $\phi(\rho_0) = 0$. In contrast, based on Wngsness [106, p.57, Figure 3-8], we can derive Wngsness [106, p.75, (5-32)] and find $\rho_0 = (4L_2L_1)^{1/2}$. Both Wngsness [106, p.75, (5-28)] and Wngsness [106, p.75, (5-32)] lead to $\phi(\infty) = \infty$. However, the latter method uses Wngsness [106, p.75, (5-31)] to get a better estimate, so we can gain better information about $\rho_0$. See Wngsness [106, p.76, l.6–l.9].

Example 6.61. (How to rigorously prove an intuitive statement)

The illustration given in Wngsness [106, p.237, l.1–l.2] at best provides the idea of proof instead of a detailed proof. In order to highlight the key idea and provide a rigorous proof, we should simplify our model and make it typical so that we may easily generalize this special case to the general case. In other words, the following factors must be simplified: the shape of $C'$, the positions of $P$ and $P + ds$, and the solid angles. Considering symmetry and the simplification of solid angles, we let $C'$ be $|r'| = a$ [where $r' = (x,y,0)$], $P = (x,y,z)$ be on the positive $z$-axis, and $P' = P + (0,0,dz)$, where $dz > 0$. Let $T_0$ be the right circular cone with vertex $P$ and base $C'$, $T_1$ be the right circular cone with vertex $P'$ and base $C''$, and $T_2$ be the right circular cone with vertex $P$ and base $C' + (0,0,-dz)$. Then $T_1 \cong T_2$. Therefore, the solid angle $\Omega'$ subtended at the vertex of $T_1$ equals to the solid angle subtended at the vertex of $T_2$.

Example 6.62. (Somewhat indirect calculations vs. direct calculations [Wngsness [106, p.65, 1.9–1.12; p.248, 1.2–1.4]]

I. (a). Somewhat indirect calculation of $\nabla \cdot \mathbf{E}$:

The geometric operation that transforms $\int_S \mathbf{E} \cdot d\mathbf{a}$ to a solid-angle integral:
Wngsness [106, p.58, (4-2) → (4-1)] by using Wngsness [106, p.58, (4-3) & (4.4)].

Calculus operations: (i). Express $Q_m$ as a volume integral [Wngsness [106, p.60, l.15]].
(ii). Use the divergence theorem to transform $\int_S \mathbf{E} \cdot d\mathbf{a}$ to a volume integral [Wngsness [106, p.60, the left-hand side of (4-9)]].

(b). Direct calculation of $\nabla \cdot \mathbf{E}$:

Express $\nabla \cdot \mathbf{E}$ in a form good for calculations:
Wngsness [106, p.65, (4-22) → (4-23) → (4-24) → p.66, (4-24)].

The geometric operation that transforms the integral given in the right-hand side of Wngsness [106, p.66, (4-25)] to a solid-angle integral: Wngsness [106, p.66, (4-25) → (4-26)].

(c). Thus, from the viewpoint of argument structures, method I.(a) and method I.(b) are almost the same except the order of operations. However, since method I.(a) essentially considers the volume integral of $\nabla \cdot \mathbf{E}$, while method I.(b) directly considers $\nabla \cdot \mathbf{E}$, the latter method is more direct.

II. (a). Somewhat indirect calculation of $\nabla \times \mathbf{B}$:

The geometric operation that transforms $\int_S \mathbf{B} \cdot d\mathbf{s}$ to a solid-angle integral:

Calculus operations: (i). Express $I_m$ as a surface integral [Wngsness [106, p.204, (12-6)]].
(ii). Use Stokes’ theorem to transform $\int_S \mathbf{B} \cdot d\mathbf{s}$ to a surface integral [Wngsness [106, p.60, the left-hand side of (4-9)]].

(b). Direct calculation of $\nabla \times \mathbf{B}$:

Express $\nabla \times \mathbf{B}$ in a form good for calculations:

The geometric operation that transforms the integral given in the right-hand side of Wngsness [106, p.248,
(15-34)] to a solid-angle integral: Wangsness [106, p.248, (15-34) → p.248, 1.—2].

c. Thus, from the viewpoint of argument structures, method II.(a) and method II.(b) are almost the same except the order of operations. However, since method II.(a) essentially considers the surface integral of $\nabla \times \mathbf{B}$, while method II.(b) directly considers $\nabla \times \mathbf{B}$, the latter method is more direct.

**Example 6.63.** (The central-force method vs. the Coriolis method [Marion–Thornton [70, p.399, l.10–p.404, l.9]])

A. (Find the horizontal deflection by the plumb line caused by the Coriolis force acting on a particle falling freely from a height)

I. (The central-force method) This method applies the entire formalism (General theory of central-force motion) to a specific problem:

a. Derivation of the equation of motion (ellipse):

Lagrangian [Marion–Thornton [70, p.289, (8.7)]
→ angular momentum [Marion–Thornton [70, p.290, (8.10)]
→ energy [Marion–Thornton [70, total energy: p.291, (8.14); potential energy: p.297, (8.37)]
→ the equation of motion (part of an ellipse) [Marion–Thornton [70, (8.15) → (8.17) = (8.38) → (8.41)].

Note that the coefficients [Marion–Thornton [70, p.300, (8.40)] of Marion–Thornton [70, p.300, (8.41)] are quite complicated. The $\theta$ given in Marion–Thornton [70, p.402, (10.37)] is the complement angle of the $\theta$ given in Marion–Thornton [70, p.300, (8.41)] [Symon [99, p.131, Fig. 3.38; (3.244)]; Marion–Thornton [70, p.403, Figure 10-10]].

b. Analysis of the ellipse:

Eccentricity: $e \simeq 1$ [Marion–Thornton [70, p.402, l.7]] by Marion–Thornton [70, p.300, (8.40)]. Let $k = \frac{GM}{E}$.

One focus is close to the center of Earth [Marion–Thornton [70, p.402, l.7]]. This statement follows from [Marion–Thornton [70, p.402, l(10.38)] (let $\theta = \pi$).

c. Express $t$ in terms of $\theta$ [Marion–Thornton [70, p.402, (10.39)]] using Marion–Thornton [70, p.290, (8.12)].

Express $T$ in terms of $\theta_0$ [Marion–Thornton [70, p.404, l.1–l.10]] using Marion–Thornton [70, p.403, (10.41)].

Compare the amount that the particle is deflected eastward with the amount that the point on Earth directly beneath the initial position of the particle moves eastward at time $T$ [Marion–Thornton [70, p.404, l.13]].

II. (The Coriolis method) This method is tailored to the problem’s needs by following strategy:

a. Consider $g$ [Marion–Thornton [70, p.396, (10.33)]] instead of $g_0$ [Marion–Thornton [70, p.397, Figure 10-6]] so that Marion–Thornton [70, p.396, (10.32)] can be reduced to Marion–Thornton [70, p.397, (10.34)]. This way allows us to separate the Coriolis force from the centrifugal force and focus directly on the Coriolis force.

b. Treat the fictious centrifugal force and Coriolis force as if they were real forces. This way allows us to have a simple rectangular coordinate system as in Marion–Thornton [70, p.400, Figure 10-9], and to work in a rotating frame as if it were a fixed frame. The use of this coordinate system greatly reduces the calculations [Marion–Thornton [70, p.400, l.1–p.401, l.15]].

c. This method uses only Marion–Thornton [70, p.397, (10.34)]. We need not consider any concept given in I.a.

B. The following two articles provide the proofs of Euler’s rotation theorem (Express a rotation in terms of the direction of axis of rotation and the angle of rotation):


The matrix proof given in the first article shows only that the direction of axis of rotation exists. The existence is provided by the reduction to absurdity. That is, we cannot point out which diameter is the desired one. However, this drawback can be remedied by solving a system of linear equations.

Let us compare the geometric proof given in Article 1 with the linear-algebra proof given in pp.1–3 of Article 2. At first glance, the latter proof seems simpler than the former one. However, the argument given in p.2, l.−4–p.3, l.10 of Article 2 uses Jacobson [56, vol.2, p.185, Theorem 8] and Rudin [88, p.79, Definition 4.1]. Therefore, the former proof is simpler than the latter one. This is because the former proof tailors to the problem’s needs, while the latter proof is based on the general theory of linear algebra, and thus involves too broad an area for consideration.

Euler’s method of tailoring a solution to the problem’s needs: He first assumes the solution (the fixed point O on the sphere by the given rotation) exists. Then he finds its consequent property (the great circle OA bisects ∠αAa). This observation allows him to conclude that a fixed point by rotation must have this property. Therefore, he first bisects ∠αAa with a great circle. Then he attempts to look for the fixed point along the circle. Thus, he reduces his search scope from the entire sphere to a particle circle.

In Landau–Lifshitz [62, p.110, l.3–l.13], Euler attempts to rotate the axes of the fixed space frame to the corresponding axes of the moving frame [Landau–Lifshitz [62, p.110, l.4–l.8]] by the same method. He first assumes that the solution exists. Then he analyzes its properties [Landau–Lifshitz [62, p.110, Fig. 41; l.3–l.11]]. Finally, he builds the required rotations based on these properties [Landau–Lifshitz [62, p.110, l.10–l.15]]; the key lies in the fact that ˙φ, ˙θ, and ˙ψ are linearly independent.

It is not clear how the author reaches the conclusion given in p.3, l.8–l.10 of Article 2. Furthermore, in the subsection of the axis-angle representation in §4.2, he abandons all the results he has achieved in p.1–p.3 of Article 2 and starts from scratch to derive formula (5) in p.6 of Article 2. This is unnecessary. In fact, it suffices to prove Rodrigues’ formula when

\[
R = \begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Rodrigues’ formula allows us to directly read off the direction of axis of rotation and the angle of rotation.

Now let us fill the gap in p.3, l.8–l.10 of Article 2.

**Proof.** Based on Rudin [88, p.79, Definition 4.1], we can prove \((\bar{x}, \bar{x}) = (x, x), ||v_1|| = ||v_2||, (v_1, v_2) = 0.\) Thus, we may let \(v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\) without loss of generality. Suppose \(v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\).

\[
\begin{bmatrix}
R[1]
\end{bmatrix} = \begin{bmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix}
1 \\ 0 \\ 1
\end{bmatrix} = \begin{bmatrix}
\cos \phi \\
\sin \phi
\end{bmatrix}.
\]

Therefore, \(R = \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}\). 

Conclusion: In most cases, we should concentrate on a small area and tailor a solution to problem’s needs when we try to discover or prove a unproven statement, find the origin for its discovery or the incentive for its proof, or look for the key idea behind the proof. In contrast, we should use the general theory to prove a specific theorem if the theorem is already proven and we want to see what role the theorem plays in the general theory from hindsight. In addition, if we divide the general theory into several categories, we would like to see to what category the theorem belongs for classification.

C. The angular velocity of a rigid body [Landau–Lifshitz [62, p.97, l.−5–p.98, l.2]] refers to the angular

Remark 1. The change resulting from the infinitesimal rotation \( \delta \vec{\theta} \) [Marion–Thornton [70] p.35, (1.106); p.36, Figure 1-19] can be expressed in matrix form:

\[
\delta \vec{r} = \delta \vec{\theta} \times \vec{r} \quad \text{(Marion–Thornton [70], p.35, (1.106))}
\]

\[
\begin{bmatrix}
\delta \theta_1 \\
\delta \theta_2 \\
\delta \theta_3 \\
\end{bmatrix}
\begin{bmatrix}
0 & -\delta \theta_3 & -\delta \theta_2 \\
\delta \theta_3 & 0 & -\delta \theta_1 \\
-\delta \theta_2 & \delta \theta_1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\]

Remark 2. Why can we neglect higher-order infinitesimals in Goldstein–Poole–Safko [41] p.165, (4.67)?

**Explanation.** The general infinitesimal rotation \( I + [\varepsilon_{ij}] \) can be represented by

\[
A = d\vec{\psi}d\vec{\theta}d\vec{\phi} =
\begin{bmatrix}
1 & (d\phi + d\nu) & 0 \\
-(d\phi + d\nu) & 1 & d\theta \\
0 & -d\theta & 1 \\
\end{bmatrix}
\]

[Goldstein–Poole–Safko [41] p.165, 1.14].

Namely, \( d\vec{\Omega} = id\theta + k(d\phi + d\nu) \) [Goldstein–Poole–Safko [41] p.165, 1.15].

Our goal is to divide \( \delta \vec{r} = \delta \vec{\theta} \times \vec{r} \) [Marion–Thornton [70] p.35, (1.106)] by \( dt \) and then let \( dt \to 0 \) in order to obtain \( \vec{v} = \vec{\omega} \times \vec{r} \) [Marion–Thornton [70] p.35, (1.105)].

We keep the first-order infinitesimals because only the first-order limits \( \lim_{dt \to 0} \frac{d\phi}{dt}, \lim_{dt \to 0} \frac{d\theta}{dt}, \lim_{dt \to 0} \frac{d\nu}{dt} \) can be nonzero. In contrast, the second-order limits \( \lim_{dt \to 0} \frac{d\phi d\theta}{dt^2}, \lim_{dt \to 0} \frac{d\theta d\nu}{dt^2} \) must be zero. Consequently, we may ignore the higher-order infinitesimals.

Remark 3. The general infinitesimal rotation given by Goldstein–Poole–Safko [41] p.165, 1.14] is a matrix, so it is necessary to prove it to be a vector [i.e. show the independence of order for their composition function]. Goldstein–Poole–Safko [41] p.165, (4.67)] provides the proof. In Marion–Thornton [70] p.35, (1.106), the general infinitesimal rotation is defined by the vector \( d\vec{\theta} \) that satisfies \( \delta \vec{r} = \delta \vec{\theta} \times \vec{r} \). Thus, \( d\vec{\theta} \) is already a vector whose magnitude and direction are the same as those in Landau–Lifshitz [62] p.18, 1.–9–1.8. Consequently, the proof given in Marion–Thornton [70] p.36, 1.8–l.–l.1] is totally unnecessary and the statement given in Marion–Thornton [70] p.36, 1.3–l.4] is not true. From the viewpoint of vectors, we understand that two velocities applied to the same position can be added. It is why two angular velocities can be added that requires an explanation. This is the main problem whose answer is provided by the distributive law for vectors. As for the problem of showing the independence of order for their composition function [Marion–Thornton [70] p.36, 1.4–l.7]; Goldstein–Poole–Safko [41] p.165, (4.67)] is only a side problem. The key to proving that the composition function for \( d\vec{\theta} \) is independent of order is also the distributive law for vectors. See Marion–Thornton [70] p.36, 1.8–l.–l.1]. Actually, the proof given in Marion–Thornton [70] p.36, 1.8–l.–l.1] is essentially the same as the proof given in Symon [99] p.452, 1.–l.15–l.–7] except that the summations in the former proof are infinitesimal rotations, while the summations in the latter proof are not restricted to infinitesimal angular velocities.
In my opinion, the tool of matrix is good for describing the result of motion, but is not appropriate for describing the process of motion; for example, angular velocities.

**Example 6.64.** (Formal methods vs. heuristic methods)

Lagrange’s equation of motion does not have a definite physical meaning because it can refer to 
\( F = ma \) [p. VI-1, (6.4) of http://www.people.fas.harvard.edu/~djmorin/chap6.pdf] or \( d\mathbf{M}/dt = \mathbf{K} \) [Landau–Lifshitz 62 p.108, (34.3); p.109, 1.13–1.14]]. When we apply it to a practical problem, we simply substitute the data into the equation without considering its derivation. Thus, a formal method puts physical meanings in a black box and plays with mathematical formulas alone. A formal proof makes it difficult for us to see the motivation behind it.

I. The component of angular momentum along an axis about which the field is symmetrical is always conserved [Landau–Lifshitz 62 p.20, l.−5–l.−3].

**Heuristic proof.** Let z-axis be the symmetrical axis of the field. 
\( \mathbf{K} = \mathbf{r} \times \mathbf{F} \), so \( \mathbf{K}_z = 0. \)

\[ d\mathbf{M}/dt = \mathbf{K} \Rightarrow d\mathbf{M}_z/dt = 0. \]

**Formal proof.** 
\[ \frac{\partial \mathbf{L}}{\partial \dot{\phi}} - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = 0 \] [Marion–Thornton 70 p.238, (7.18)].

\[ \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = \mathbf{M}_z \] [Landau–Lifshitz 62 p.21, (9.7)].

By Wangsness [106 p.30, (1-86)], \( U = U(\rho) \). Consequently, \( \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = -\frac{\partial U}{\partial \phi} = 0. \)

II. The component of angular momentum along any axis through the centre is conserved in motion in a central field [Landau–Lifshitz 62 p.21, l.3–l.5].

**Heuristic proof.** See Symon 99 p.123, l.−13–l.−11].

**Formal proof.** 
\[ \frac{\partial \mathbf{L}}{\partial \dot{\phi}} - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = 0 \] [Marion–Thornton 70 p.238, (7.18)].

\[ \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = \mathbf{M}_z \] [Landau–Lifshitz 62 p.21, (9.7)].

For a central field, \( U = U(r) \) [Landau–Lifshitz 62 p.21, l.2–l.3]. Consequently, \( \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = -\frac{\partial U}{\partial \phi} = 0. \)

III. The component \( \mathbf{M}_z \) of the angular momentum is conserved [Landau–Lifshitz 62 p.21, l.8]].

**Heuristic proof.** \( \mathbf{K} = \mathbf{r} \times \mathbf{F} \), so \( \mathbf{K} \) is perpendicular to the z-axis.

\[ d\mathbf{M}/dt = \mathbf{K} \Rightarrow d\mathbf{M}_z/dt = \mathbf{K}_z = 0. \]

**Formal proof.** 
\[ \frac{\partial \mathbf{L}}{\partial \dot{\phi}} - \frac{d}{dt} \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = 0 \] [Marion–Thornton 70 p.238, (7.18)].

\[ \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = \mathbf{M}_z \] [Landau–Lifshitz 62 p.21, (9.7)].

By Wangsness [106 p.30, (1-86)], \( U = U(z) \). Consequently, \( \frac{\partial \mathbf{L}}{\partial \dot{\phi}} = -\frac{\partial U}{\partial \phi} = 0. \)

Remark. Although the Lagrangian formalism is ineffective in local view, it is useful in global view. For example, it provides the mathematical foundation of the analogy between the two columns at the bottom of Symon 99 p.211]. Thus, it unifies the theory of rectilinear motion and the theory of rotation about a fixed axis. If we know the former theory alone, we have to use the analogy as a guide to study the latter theory.
Example 6.65. (A theory that leads to a contradiction can still be useful)

The theory of quasi-static approximation [Choudhury [20, p.254, (6.25)]) leads to a contradiction: the velocity of propagation of the quasi-static electromagnetic waves is infinite [Choudhury [20, p.255, l.15–l.14]). The formulas given in Choudhury [20, p.254, (6.25)] should be corrected as in Choudhury [20, p.254, (1.18)]. However, the quasi-static approximation is simpler than the correct theory about Maxwell’s equations and has wide applications [Choudhury [20 §6.4–§6.7]]

Because of the above contradiction, the quasi-static approximation can only be applied in those cases where the time lag produced by the finite velocity of propagation of quasi-static electromagnetic waves is negligible [Choudhury [20, p.255, l.18–l.20]. Namely, the approximation is only good for small regions [Choudhury [20 p.255, l.13–l.11]] and low frequencies [Choudhury [20 p.255, (6.31)]. Thus, we consider only electromagnetic fields inside a region compatible with the quasi-static approximation [Choudhury [20 p.260, l.6–l.7]].

Example 6.66. (Classical derivation of the macroscopic Maxwell equations)

Choudhury [20 §7.2] derives the macroscopic Maxwell equations from the microscopic Maxwell equations. The proof uses test functions rather than probabilities in quantum mechanics to define the concept of average. Thus, the approach considers quantum mechanics as a black box and fails to accurately indicate how the inner structure of quantum mechanics works in this case. Let us establish a closer relationship between the derivation and quantum mechanics.

Example 6.67. (Energies and forces under various conditions)

I. In vacuum

A.
The electric energy of an isolated system of a given charge distribution [Wangsness [106, p.101, l.−14–l.−10]]

a. The formula for energy: When using Wangsness [106, p.102, (7-28)], we should know

1. Its derivation: In order to express the energy in terms of fields [Wangsness [106, p.101, l.−10–l.−9]], we start with Wangsness [106, p.99, (7-10); p.60, (4-10)].

Process of derivation: Wangsness [106, (7-1)→(7-6)→((7-7), (7-8), (7-9), (7-10))→(7-22)→(7-25)→(7-28)].

2. The charges can be free charges or bound charges [Wangsness [106, p.161, l.6–l.10]].

b. Properties:

1. Wangsness [106, p.101, (7-21); p.102, (7-28)] can be used to calculate capacitance.

2. When the total charge is fixed, different charge distributions lead to different energies [Compare Wangsness [106, p.100, (7-14)] with Wangsness [106, p.101, (7-20)].

3. The energy of a capacitor given in Wangsness [106, p.101, (7-21)] can also be obtained by using only general properties of work and potential difference as in Wangsness [106, p.108, Exercise 7-3].

The magnetic energy of an isolated system of a given current distribution [Wangsness [106, p.287, l.5–l.8]]

a. The formula for energy: When using Wangsness [106, p.287, (18-21)], we should know

1. Its derivation: In order to express the energy in terms of the magnetic induction [Wangsness [106, p.287, l.8–l.9]], we start with Wangsness [106, p.286, (18-12); p.241, (15-12)].


b. Properties:

1. Wangsness [106, p.286, (18-9); p.287, (18-21)] can be used to calculate inductance.

Remark. \(dU_m\) in Wangsness [106, p.284, (18-2)] refers to the middle stage of process, while \(dU_m\) in Wangsness [106, p.291, l.12] refers to the final stage of process. Therefore, they are different [Wangsness [106, p.291, l.14–l.15]].
The electric force on the positive plate of a parallel capacitor [Wangsness [106] p.104, Figure 7-1]

- a. Constant charges
  \[ F_e = -(\frac{dU_e}{dx})Q \]  
  [Wangsness [106] p.105, (7-37)].
- b. \( \Delta \phi = \text{const.} \)
  \[ F_e = + (\frac{dU_e}{dx}) \Delta \phi \]  
  [Wangsness [106] p.106, (7-45)].

Both possible conditions lead to the same result [Wangsness [106] p.105, (7-39); p.106, (7-46)].

The electric force derived from energy changes [Wangsness [106] p.108, (7-52)] agrees with the electric force derived from electric fields. See eq.1 of https://www.emworks.com/application/force-on-a-capacitor

Oppositely charged plates attract each other [Wangsness [106] p.107, l.1–l.2]

The magnetic force between two circuits [Wangsness [106] p.218, Figure 13-1]

- a. Constant currents
  \[ F_m = (\nabla U_m)_I \]  
- b. Constant flux
  \[ F_m = -(\nabla U_m) \phi \]  

Both possible conditions lead to the same result [Wangsness [106] p.291, l.−10; p.292, −16].


Like currents attract each other; unlike currents repel each other [Wangsness [106] p.293, l.−17–l.−1; p.295, l.13–l.14].

Remark 1. In order to derive forces from energy changes, we need to know the concept of static equilibrium [Wangsness [106] p.104, l.16; p.290, l.−16] and that of reversible process Wangsness [106, p.104, l.−12; p.290, l.−8]. For static equilibrium, see http://www.physicsclassroom.com/class/estatics/Lesson-3/Newton-s-Laws-and-the-Electrical-Force. For reversible process, see Kittel [59, p.64, l.2–l.3].

Remark 2. When attempting to find the force from energy change, we often question about the direction of the force. See Wangsness [106] p.293, l.19]. The explanation given in Wangsness [106] p.293, l.19–l.−1 is quite long and fails to hit the heart of the matter. This is because Wangsness [106] p.293, l.15] uses Wangsness [106, p.291, (18-41)] instead of Wangsness [106, p.291, (18-40)], and thus loses the information about direction. Consequently, the correct answer should lie in vector analysis. If we use Wangsness [106, p.291, (18-40)] instead, the solution would become simple:

\[
\mathbf{F}_m = I' \nabla M \]  

\[
= I' \left[ \frac{\partial M}{\partial x}, \frac{\partial M}{\partial y}, \frac{\partial M}{\partial z} \right] 
= I' \left[ \frac{\partial M}{\partial x} \hat{x} + \frac{\partial M}{\partial y} \hat{y} + \frac{\partial M}{\partial z} \hat{z} \right] 
= I' \frac{\partial M}{\partial x} \hat{x} \quad \text{[by Wangsness [106] p.293, l.15], } M \text{ is independent of } y, z. 
\]

Remark 2’. Wangsness [106] p.107, l.1–l.2; p.107, l.−8–p.108, l.1] attempt to explain how we determine the direction of electric force, but fail to hit the heart of the matter. Let us solve this problem using vector analysis instead.

\[
dU_e = dU_t \quad \text{[Wangsness [106] p.12, (1-38)]} 
= -F_e \cdot dr \quad \text{where } F_e = (F_e, 0, 0). \text{ Up to now } F_e \text{ is only a vector whose components are assigned.} 
F_e = -\nabla U_e \quad \text{[Wangsness [106] p.105, l.17]; this step shows that } F_e \text{ is the desired electric force]}
= -\left[ \frac{\partial U_e}{\partial x}, \frac{\partial U_e}{\partial y}, \frac{\partial U_e}{\partial z} \right] \]
Remark 3. By Wangsness [106, p.259, (16-48)], $\Phi = \mu_0 n \pi \rho^2 I$. Thus, the sign of $\Phi$ is the same as that of $I$. Therefore, the statement $\Phi > 0$ given in Wangsness [106, p.293, l.20–l.21] is incorrect. However, $M$ and $\frac{\partial M}{\partial x}$ are positive.

Proof. By Wangsness [106, p.278, (17-45)], $\Phi_{I \rightarrow I'} = MI$.
Consequently, $M = \mu_0 n \pi \rho^2 > 0$.
By Wangsness [106, p.293, l.15], $M = \mu_0 n n' S x$.
Hence $\frac{\partial M}{\partial x} = \mu_0 n n' S > 0$. \qed

II. (The interaction energy of the system group in the field produced by the external group) The sources are divided into groups: the system group and the external group. The two groups are some distance apart and are thus easily distinguishable. We disregard the internal energy of each of the two groups and are interested in only the interaction energy of the source distribution of the system group in the field produced by the external group.
Electric energy of a charge distribution in an external electric field [Wangsness 106 §8.4]

The total electric energy can be found by Wangsness 106 p.80, (5-48). The part of electric energy in which we are interested is the interaction energy given by Wangsness 106 p.124, (8-60).

If the external sources are far away and the system is small [Wangsness 106 p.124, 1–5–1–1], then we have the multipolar expansion for the interaction energy: Wangsness 106 p.126, (8-71); p.127, (8-72).

The dipole energy in an external electric field
\[ U_D = -p \cdot E_0 \] [Wangsness 106, p.127, (8-73)]
\[ = -pE_D \cos \psi. \]
\[ U_I = U_D \] [Wangsness 106, p.105, 1.17].
\[ \tau = -dU_I/d\psi = pE_D \sin \psi \] [We should use this formula to correct Wangsness 106 p.127, (8-74)].
\[ \tau \cdot d\psi = \tau d\psi = pE_0 \sin \psi d\psi = (p \times E_0) \cdot \psi. \]
By Rudin 88 p.31, Theorem 1.39(b),
\[ \tau = p \times E_0. \]
The direction \( p \times E_0 \) turns out to be the most effective direction to decrease the total energy \( U_I \) (i.e. to decrease \( \psi \) in Wangsness 106 p.128, Figure 8-11).
\[ F_D = -\nabla U_I \] [Wangsness 106 p.104, (7-36)]
\[ = -\nabla U_D \] [Wangsness 106, p.105, 1.17]
\[ = \nabla(p \cdot E_0) \] [Wangsness 106 p.128, (8-77)]
\[ = (p \cdot \nabla)E_0 \] [Wangsness 106 p.129, (8-79)].
Wangsness 106 p.129, 1.10–p.130, 1.3 shows that the force [Wangsness 106 p.129, (8-79)] and torque [Wangsness 106 p.127, (8-75)] derived from the energy change agree with those derived from the potential and electric field in the point dipole case.

Magnetic energy of a current distribution in an external magnetic induction [Wangsness 106 §19.4]

The total magnetic energy: Wangsness 106 p.285, (18-8). The part of magnetic energy in which we are interested is the interaction energy given by Wangsness 106 p.305, (19-31), (19-32); p.306, (19-33), (19-34), (19-35).

If the external sources are far away and the system is small [Wangsness 106 p.306, 1.15–1.17], then the first approximation of Wangsness 106 p.306, (19-33) will be Wangsness 106 p.306, (19-36).

The dipole energy in an external magnetic induction
\[ U_{m0D} = m \cdot B_0(r) \] [Wangsness 106 p.306, (19-36)]
\[ U_I = -U_{m0D} \] [Wangsness 106 p.291, 1.19]
\[ = U_D' \] [Wangsness 106 p.307, (19-40)]
\[ \tau = -dU_D'/d\psi = -mB_0 \sin \psi. \]
\[ \tau \cdot d\psi = \tau d\psi = mB_0 \sin \psi d\psi = (m \times B_0) \cdot \psi. \]
By Rudin 88 p.31, Theorem 1.39(b),
\[ \tau = m \times B_0. \]
The direction \( m \times B_0 \) turns out to be the most effective direction to decrease the total energy \( U_D' \) (i.e. to decrease \( \psi \) in Wangsness 106 p.307, Figure 19-8).
\[ F_D = -\nabla U_I = -\nabla U_D' \]
\[ = \nabla(m \cdot B_0) \] [Wangsness 106 p.307, l.–9–l.–8]
\[ = (m \cdot \nabla)B_0 \] [Wangsness 106 p.307, (19-39)].


Remark 1.

a. \( F_e = -dU_I/dx \) [Wangsness 106 p.104, (7-36)].
Its derivation: \( dU_I = F_{\text{mech}} dx, F_e = -F_{\text{mech}} \) [Wangsness 106 p.104, l.–10; l.–8].
The formula implies that the direction of electric force is the most effective direction to decrease the total energy.
\[ \tau_e = -\frac{\partial U_t}{\partial \psi} \text{ [Wangsness [106, p.127, (8-74)]].} \]

Its derivation: \(dU_t = \tau_{\text{mech}} d\psi, \tau_e = -\tau_{\text{mech}}\) [Wangsness [106, p.128, (8-76)]].

The formula implies that the direction of electric torque is the most effective direction to decrease the total energy.

b. \(F_m = -\nabla U_t\) [Wangsness [106, p.290, (18-36)]].

Its derivation: \(dU_t = F_{\text{mech}} \cdot dr, F_m = -F_{\text{mech}}\) [Wangsness [106, p.290, l.-7; l.-5]].

The formula implies that the direction of magnetic force is the most effective direction to decrease the total energy.

c. \(F_G = -\frac{dU_t}{dh}\).

Its derivation: \(dU_t = F_{\text{mech}} dh, F_G = -F_{\text{mech}}\)

The formula implies that the direction of gravitational force is the most effective direction to decrease the total energy.

Remark. In the above table, I use \(F_D = -\nabla U_D, \tau = -\frac{dU_D}{d\psi}\) rather than \(F_D = \nabla U_{m0D}, \tau = \frac{dU_{m0D}}{d\psi}\) [Wangsness [106, p.306, (19-37)]] because the former pair has the same physical interpretation as the corresponding pair in the electric field.

c. \(F_G = -\frac{dU_t}{dh}\).

Its derivation: \(dU_t = F_{\text{mech}} dh, F_G = -F_{\text{mech}}\)

The formula implies that the direction of gravitational force is the most effective direction to decrease the total energy.

Remark. The \(F\) in Marion–Thornton [70, p.78, (2.87)] refers to \(F_G\); the \(F\) in Marion–Thornton [70] p.78, (2.84)] refers to \(F_{\text{mech}}\). Thus, Marion–Thornton [70] §2.5] uses the same symbol \(F\) to represent two opposite forces. Furthermore, Marion–Thornton [70] §2.5] fails to point out that the potential energy is the interaction energy of the particle in the gravitational field. These show how unprepared our current textbooks in classical mechanics are for the study of electromagnetic fields. Marion–Thornton [70, p.79, (2.88)] can be proved by Wangsness [106, p.12, (1-38)].

Remark 2. The idea given in Wangsness [106, p.127, 1.–6–l.–1] is good, but the proof of Wangsness [106, p.127, (8-75)] is incorrect. Thus, a good idea requires good skills to carry it out.

III. A.
The electric energy for a system of free charges in the presence of matter

| a. The formula for energy: When using Wangsness [106] p.161, (10-78); p.162, (10-84), we should have the following things in mind:
| b. Properties: Note that the deduction method given in Wangsness [106, p.80, (5-48)]. |
| 1. The electric energy of the system should be the retrievable one contributed by free charges that we can control [Wangsness [106, p.161, l.10–l.18]]. |
| 2. Its derivation: In order to fit in the situation that the system is in, we start with Wangsness [106] p.80, (5-48)]. |

Note that the deduction method given in Wangsness [106] (10-82)] is the same as that given in Wangsness [106] (7-22)→(7-28)] in I.A.case of electric energy.a.1. 

b. Properties:

1. The presence of dielectric changes $D,E$, and hence $U_e$ in Wangsness [106] p.162, (10-84)]. The exact amount of change will depend on the particular manner in which the process is carried out [Wangsness [106] p.163, l.19]]. 

(i). The case that we can ascribe the energy change to the dielectric itself Given a charge distribution in vacuum with resulting fields $D_0$ and $E_0$ $\rightarrow$ the energy change due to the bound charges of the dielectric [Wangsness [106] p.163, (10-91)]] $\rightarrow$ Wangsness [106] p.164, (10-92)]. 

Proof. $\int_{\text{all space}} (E \cdot D - E_0 \cdot D_0) d\tau = \int_{\text{all space}} (E \cdot D_0 - E \cdot E_0) d\tau + \int_{\text{all space}} (E + E_0) \cdot (D - D_0) d\tau.$ $\int_{\text{all space}} (E + E_0) \cdot (D - D_0) d\tau = -\int_{\text{all space}} \nabla \phi \cdot (D - D_0) d\tau [\nabla \times (E + E_0) = 0 \Rightarrow E + E_0 = -\nabla \phi] = \int_{\text{all space}} \phi \nabla \times (D - D_0) d\tau [Wangsness [106] p.34, (1-115)].$ $\int_{\text{all space}} \phi \nabla \times (D - D_0) d\tau = 0 [\nabla \cdot D = \rho = \nabla \cdot D_0].$ $U_{eb} = -\frac{1}{2} \int_V (\varepsilon - \varepsilon_0) E \cdot E_0 d\tau [D = \varepsilon_0 E$ outside $V] = -\frac{1}{2} \int_V \varepsilon_p \cdot E_0 d\tau [Wangsness [106] p.151, (10-40)].$ 

The magnetic energy for a system of free currents in the presence of matter

| a. The formula for energy: When using Wangsness [106] p.333, (20-74); p.334, (20-81)], we should have the following things in mind:
| b. Properties: Note that the deduction method given in Wangsness [106, p.335, l.16–l.17]]. |
| 1. The magnetic energy of the system should be the retrievable one contributed by free currents that we can control [Wangsness [106] p.335, l.16–l.17]]. |
| 2. Its derivation: In order to fit in the situation that the system is in, we start with Wangsness [106] p.333, (20-74)]. |
| b. Properties:

1. The presence of magnetic will generally change $H,B$, and hence $U_m$ in Wangsness [106] p.334, (20-81)]. The exact amount of change will depend on the process by which the matter is introduced into the field, and a general discussion can be quite complex [Wangsness [106] p.335, (20-88)]] $\rightarrow$ Wangsness [106] p.335, (20-89)]].

Proof. $\int_{\text{all space}} (B \cdot H - B_0 \cdot H_0) d\tau = \int_{\text{all space}} (B \cdot H_0 - H \cdot B_0) d\tau + \int_{\text{all space}} (B + B_0) \cdot (H - H_0) d\tau.$ $\int_{\text{all space}} (B + B_0) \cdot (H - H_0) d\tau = \int_{\text{all space}} \nabla \times A \cdot (H - H_0) d\tau [\nabla \cdot (B + B_0) = 0 \Rightarrow E + E_0 = \nabla \times A] =$ $\int_{\text{all space}} A \cdot [\nabla \times (H - H_0)] d\tau [Wangsness [106] p.34, (1-116)].$ $0 [\nabla \times H = J = \nabla \times H_0].$ $U_{mm} = -\frac{1}{2} \int_V (\frac{\mu_0}{\rho} - H) \cdot B_0 d\tau [H = \frac{\mu_0}{\rho}$ outside $V; H_0 = \frac{\mu_0}{\rho}];$ $\Rightarrow \frac{1}{2} \int_V M \cdot B_0 d\tau [Wangsness [106] p.332, (20-28)].$ $\square$
The electric energy for a system of free charges in the presence of matter

The $U_{eb}$ in Wangsness [106, p.164, (10-92)] can be considered the internal energy of the dielectric system [Wangsness [106, p.164, 1.5–1.6]]. By comparing Wangsness [106, p.164, (10-92)] with Wangsness [106, p.164, (10-95)], we see that the internal energy $U_{eb}$ is different from the interaction energy $U_{e, ext}$.

(ii). The effect of a dielectric on a capacitor:

$\alpha$. If $Q$ is kept constant, $U_e = U_0/\kappa_e$ [Wangsness [106, p.164, (10-96)]] and $C = \kappa_e C_0$ [Wangsness [106, p.159, (10-73)]].

$\beta$. If $\Delta \phi$ is kept constant, $U_e = \kappa_e U_0$ [Wangsness [106, p.165, (10-97)]].

Proof of $\beta$. Wangsness [106, (10-71)–(10-72)–(10-73)–(10-97)].

The magnetic energy for a system of free currents in the presence of matter

The $U_{mm}$ in Wangsness [106, p.335, (20-89)] can be considered the internal energy of the magnetic material system [Wangsness [106, p.335, 1.2–1.3–1.4]]. By comparing Wangsness [106, p.335, (20-89)] with Wangsness [106, p.336, (20-92)], we see that the internal energy $U_{mm}$ is different from the interaction energy $U'_{m, ext}$.

(ii). The effect of a magnetic material on a coaxial line:

$\alpha$. If we keep free currents fixed, $U_{m2} = \kappa_m U_{m20}$ [Wangsness [106, p.335, (20-86)]] and $L_2 = \kappa_m L_{20}$ [Wangsness [106, p.335, 1.16]].

Remark. $U_{e1} = \frac{1}{2} \int_{\text{all space}} \rho \phi \, d\tau$ [in vacuum, Wangsness [106, p.99, (7-10)]], where $\rho$ includes free and bound charges. In contrast, $U_{e2} = \frac{1}{2} \int_{\text{all space}} \rho_f \phi \, d\tau$ [in the presence of matter, Wangsness [106, p.161, (10-78)]], where $\rho_f$ refers to free charges only. Therefore, if the system contains only free charges, then $U_{e1} = U_{e2}$. In terms of fields, this means Wangsness [106, p.102, (7-28)] = Wangsness [106, p.163, (10-90)].
The electric force of a parallel plate capacitor on a solid slab of dielectric of the right size to fit between the plates

\[
U_e = \frac{U_0}{\kappa} \quad [Q = \text{const.}] \quad \text{Wangsness [106, p.164, (10-96)]}
\]

The general tendency of the systems is to reduce their electric energy, so the capacitor will want to have the dielectric in place.

\[
\Delta U_e = -\kappa e - 1 \kappa e U_0 \quad \text{Wangsness [106, p.165, (10-98)]}
\]

\[
< F > = -\Delta U_e = \left(\frac{\kappa - 1}{\kappa} \right) U_0 \quad \text{Wangsness [106, p.166, (10-99)]}
\]

Thus, the dielectric will be pulled into the region between the plates and

\[
< f_a > = \left(\frac{\kappa - 1}{\kappa} \right) u_0 \quad \text{Wangsness [106, p.166, (10-100)]}
\]

The magnetic force of a long solenoid on a inserted permeable rod whose cross section area is the same as that of the solenoid

\[
\text{a. By Ampère's law for } H \quad \text{Wangsness [106, p.322, (20-32)]}
\]

\[
H_i = ni \hat{z} \quad \text{Wangsness [106, p.309, (20-63)]}
\]

\[
H_o = 0 \quad \text{Wangsness [106, p.309, l.-4]}
\]

\[
u_m = \frac{1}{2} \mu H \quad \text{Wangsness [106, p.334, (20-84)]}
\]

\[
U_m(z) = \frac{1}{2} \mu_0 n^2 I^2 S [\mu z + \mu_0 (l - z)] \quad \text{Wangsness [106, p.337, (20-93)]}
\]

\[
F_m = (\nabla U_m) I \quad \text{Wangsness [106, p.291, (18-39)]}
\]

\[
= \frac{1}{2} \chi_m \mu_0 n^2 I^2 S \hat{z} \quad \text{Wangsness [106, p.337, (20-94)]}
\]

If \(\chi_m > 0\), the rod will be attracted into the solenoid.

If \(\chi_m < 0\), the rod will be repelled.

b. Another qualitative derivation using the following statement:

Like currents attract; unlike currents repel.

If \(\chi_m > 0\), \(K_m, I\) have the same direction \(\text{Wangsness [106, p.318, Figure 20-7 (b)]}\), so they attract.

If \(\chi_m < 0\), \(K_m, I\) have opposite directions \(\text{Wangsness [106, p.318, Figure 20-7 (b)]}\), so they repel.

c. The conclusions to which a. and b. lead agree.

d. \(f_m = (u_m I - u_m 0) \hat{z} \quad \text{Wangsness [106, p.337, (20-96)]}\).

(i). The direction of \(f_m\) is such as to tend to move the material so as to increase the total magnetic energy of the system. For example, Assume \(\chi_m > 0\). The increase of \(U_m\) means the increase of \(z \quad \text{Wangsness [106, p.337, (20-93)]}\), so the rod will be attracted.

(ii). \(f_m = f_m I + f_m 0 \quad \text{Wangsness [106, p.337, l.-2]}\), so Wangsness [106, p.337, (20-96)] is consistent with Wangsness [106, p.295, (18-52)].

IV. Electromagnetic energy for the time varying fields

A. General case: Poynting’s theorem \(\text{Wangsness [106, (12-35)\rightarrow(21-53)\rightarrow(21-54)\rightarrow(21-55)]}\).

B. Case of l.i.h. media
\[
\oint_S (E \times H) \, da = \left( \frac{dU}{dt} \right) \text{through} \ S \quad \text{[Wangsness [106], p.357, (21-58)]},
\]

where \( S = E \times H \), the Poynting vector, is the rate of flow of electromagnetic energy per unit area.

Example. (Cylinder with constant current [Wangsness [106], p.358, Figure 21-5])

In this steady-state case, we may use the Poynting vector to show that the total rate at which energy is flowing into the volume \( S \) is equal to the total rate at which energy is being dissipated into heat within the volume \( S \).

C. a. Why we should keep the law for conservation of linear momentum rather than Newton’s third law in the presence of electromagnetic fields \([Wangsness [106], p.359, l.18–l.12]\).

b. The law for conservation of linear momentum: \([Wangsness [106], p.359, l.18–l.12]\).

D. Energy relations

\[
\langle S \rangle = \frac{1}{2} \langle (E_R \times H_R) + (E_I \times H_I) \rangle \quad \text{[Wangsness [106], p.391, (24-101)]}
\]

\[
= \frac{1}{2} \Re \langle E_C \times H_C^* \rangle \quad \text{[Wangsness [106], p.392, (24-104)]}.
\]

\[
\langle u_r \rangle = \frac{1}{2} \Re E_C \cdot E_C^* \quad \text{[Wangsness [106], p.392, (24-105)].}
\]

\[
\langle u_m \rangle = \frac{1}{4} \mu H_C \cdot H_C^* \quad \text{[Wangsness [106], p.392, (24-106)].}
\]

\[
\langle S \rangle = \langle u \rangle > \text{ for } \sigma = 0: \text{Wangsness [106], p.393, (24-111)]; } \sigma \neq 0: \text{Wangsness [106], p.393, l.15].}
\]

If \( \sigma \neq 0 \), both \( \langle S \rangle \) and \( \langle u \rangle \) are proportional to \( e^{-\beta \zeta} \) \([Wangsness [106], p.393, l.17].\]

V. A. Energy relations for reflection and transmission

a. \( R + T = 1 \) \([Wangsness [106], p.421, (25-75)].\)

b. For total reflection \( (n_1 > n_2, \theta_i > \theta_c) \), \( R = 1 \) \([Wangsness [106], p.421, l.19].\)

c. For good conductors, \( R \to 1 \) \([Wangsness [106], p.423, l.1–l.12–l.1].\)

d. Energy propagation for standing waves \([Sadiku [89], p.442, l.1–l.6–p.443, l.1–l.10]; Sadiku [89], p.450, l.5–l.11.\]

B. Radiation pressure: \( P(\theta_i) = (1 + R) \cos^2 \theta_i \langle u \rangle > \) \([Wangsness [106], p.427, (25-98)].\)

Remark. Similarity between radiation pressure and gas pressure (in kinetic theory) \([Wangsness [106], p.426, 1.6–l.49]).\)

The same figure: \([Wangsness [106], p.426, Figure 25-19] \leftrightarrow \text{Reif [84], p.280, Fig. 7.13-2}.\)

The same form: \([Wangsness [106], p.427, (25-98)] \leftrightarrow \text{Reif [84], p.281, (7.13-9)].\)

Example 6.68. (The orbit of a charged particle moving across a uniform magnetic induction is a circle \([Wangsness [106], p.533, l.10–l.11]).\)

Proof. I. \( |v(t)| = \text{const.}; |a(t)| = \text{const.}.\)

Proof. The Lorentz does no work, so the kinetic energy remains the same. Thus, \( |v| = \text{const.}.\)

\( (B \text{ is uniform}) \Rightarrow (|v \times B| \text{ const.}) \Rightarrow |a| = \text{const.} \quad \text{[Wangsness [106], p.532, (A-11)(ii)]}.\) \( \square \)

II. \( a = a_r e_r \), where \( |a_r| = \text{const.}.\)

Proof. Let \( P_1, P_2, P_3 \) be three points on the orbit,

\( v_1, v_2, v_3 \) be their corresponding velocities, and

\( a_1, a_2, a_3 \) be their corresponding accelerations. Then

The angle formed by \( v_i \) and \( v_j \) \( |i \neq j; i, j = 1, 2, 3| \).
the angle formed by $a_i$ and $a_j$ [because $a_i \perp v_i$]. Let $A_i$ be the line passing through $P_i$ and containing $a_i$. Then $A_1, A_2, A_3$ meet at the same point $G$. Thus, the desired result follows by using $G$ as the origin and then constructing a polar coordinate system on the plane containing $P_1, a_1, v_1$.

III. $(|v| = \text{const.}; |a| = \text{const.}) \Rightarrow \text{(the orbit is a circle)}$.

Proof. $|v| = \text{const.}$ [by I].

$0 = \frac{dx^2}{dt} = 2v \cdot \frac{dv}{dt}$

$\Rightarrow a \perp v$

$\Rightarrow v = v_\phi e_\phi$, where $v_\phi = \text{const.}$ [by II]

$\Rightarrow r = 0$ [Marion–Thornton \[70, p.33, (1.101)\)(iv)]

$\Rightarrow r(t) = \text{const.}$

Similarly, $r \ddot{\phi} + 2r \dot{\phi} = 0$ [Marion–Thornton \[70, p.34, l.8\]]

$\Rightarrow \dot{\phi} = 0 \Rightarrow \phi(t) = ct$ if we let $\phi(0) = 0$.

Remark. The proof given here reveals more insight with fewer calculations than that given in Wangness \[106, p.534, l.1–p.535, l.10\].

**Example 6.69.** (Solutions of Maxwell’s equations)

I. Wangness \[106, p.375, l.−9–p.376, l.3\] shows how we derive Wangness \[106, p.376, (24-7)\] from Wangness \[106, p.375, (24-1)–(24-4)\].

II. Case $\sigma = 0$:

A. The general solution of Wangness \[106, p.376, (24-9)\] is Wangness \[106, p.377, (24-11)\]. Consequently, the solutions are plane waves.

B. Wangness \[106, p.378, (24-15)\]

$\Rightarrow$ Wangness \[106, p.378, (24-16)\] [by separation of variables]

$\Rightarrow$ Wangness \[106, p.378, (24-18); p.379, (24-19)\]. Thus, the solutions are plane harmonic waves.

C. Maxwell’s equations are reduced to Wangness \[106, p.381, (24-32)\].

III. Case $\sigma \neq 0$:

We again try to solve Wangness \[106, p.376, (24-7)\] with a plane harmonic wave of the form given in Wangness \[106, p.379, (24-19)\] except that we allow $k$ to be a complex number $\alpha + i\beta$ this time. Then we obtain Wangness \[106, p.383, (24-42); (24-43)\]. Maxwell’s equations are reduced to Wangness \[106, p.381, (24-32)(i); p.385, (24-58); (24-59)\].

IV. As far as mathematical methods are concerned, these are all Wangness \[106, §24-1–§24-3\] say.

**Example 6.70.** (Boundary value problem for a vector potential)

I. (The prototype of solution) If we identify these cylinders with the appropriate equipotentials of Figure 5-8, they will carry charges of $-\lambda$ and $+\lambda$ per unit length [Wangness \[106, p.184, l.−9–l.−7\]].

Proof. Wangness \[106, p.184, Figure 11-10\] and Wangness \[106, p.77, Figure 5-8\] have the same scalar potential because of the uniqueness theorem of the solution of Poisson’s equation subject to Dirichlet bound-
ary conditions [Jackson [55, p.37, l.13–l.11]].

The desired result follows from Gauss’s law.

II. I discusses the case for a scalar potential; for a vector potential, we must replace Gauss’ law with Ampère’s law and replace the uniqueness of the solution of the Dirichlet problem with the uniqueness of vector fields in Helmboltz’s theorem [Choudhury [20, pp. 583–584]].

Theorem. Let \( A_1 = A_1(z) \) and \( A_2 = A_2(z) \). If \( A_1 \) on an \( A_z \)-equipotential line \( C \), then \( \nabla \times A_1 = \nabla \times A_2 \).

Proof. (1). \( B = \nabla A_z \times \hat{z} \).

Proof. \( \nabla A_z \times \hat{z} = - \frac{\partial A_z}{\partial \phi} \hat{\phi} + \frac{i}{\rho} \frac{\partial A_z}{\partial \theta} \hat{\theta} \) [Wangsness [106, p.30, (1-85)]]

\[ B = \text{Wangsness [106, p.31, (1-88)]}. \]

(2). By (1), \( B \)-field lines are the same as \( A_z \)-field lines.

By hypothesis, \( B_1 = B_2 \) on \( C \).

Therefore, \( A_1 \) and \( A_2 \) have the same source [by Ampère’s law].

\( \nabla \times A_1 = \nabla \times A_2 \) follows from Helmboltz’s theorem [Choudhury [20, pp. 583–584]].

III. Application. \( L' = \frac{\mu}{\pi} \cosh^{-1} \left( \frac{D}{2A} \right) \) [Wangsness [106, p.465, l.11]].

Proof. Let \( A = \frac{\rho}{\sinh \eta} \) [Wangsness [106, p.184, (11-49)] and \( \frac{D}{2} = a \cosh \eta = A \sinh \eta \) [Wangsness [106, p.184, (11-50)]. If we identify the two-wire line [radius = \( A \), distance = \( D \); Sadiku [89, p.474, Figure 11.1(b)]] with the appropriate \( A_z \)-equipotentials given in Wangsness [106, p.258, Figure 16-4], they will carry currents \(-I\) and \(+I\) [by Theorem in II]. Then \( A \) satisfies Wangsness [106, p.257, (16-37)], where \( \eta = 2\pi A_z / \mu_0 I \). Let \( R \) be a vertical rectangle whose the vertical sides pass through the points with cylindrical coordinates \((D-A, \pi, 0)\) and \((D-A, 0, 0\) respectively. Let the vertical length be \( l \). Then

\[ \Phi = \int_{R} \mu_0 I \cosh^{-1} \left( \frac{D}{2A} \right) ds \] [Wangsness [106, p.253, (16-23)]].

\[ = \frac{\mu_0 I}{2\pi} \left[ \cosh^{-1} \left( \frac{D}{2A} \right) (2l) \right] \) [by Wangsness [106, p.257, (16-35)], \( A_z(D-A, 0) = -A_z(D-A, 0) \)]

\[ = L' \int \frac{l}{l} = \frac{\mu_0 I}{\pi} \cosh^{-1} \left( \frac{D}{2A} \right). \]

Example 6.71. (Duality of electromagnetic fields)

Solve Wangsness [106, p.362, Exercise 21-12]. Show that the last two replacements in Wangsness [106, p.483, (28-77)] give an example of the duality property of electromagnetic fields found in Exercise 21-12 for they correspond to (21-77) with \( \alpha = -90^\circ \) and \( C = 1 \) [Wangsness [106, p.483, l.13–l.15]].


\[ \nabla \cdot D = 0 \quad \nabla \cdot B = 0 \]

\[ \nabla \times E = - \frac{\partial E}{\partial t} \quad \nabla \times B = \mu_0 \frac{\partial E}{\partial t}. \] Then

\[ \nabla \cdot D' = 0 \quad \nabla \cdot B' = 0 \]

\[ \nabla \times E' = - \frac{\partial E'}{\partial t} \quad \nabla \times H' = \varepsilon_0 \frac{\partial E'}{\partial t}. \]

II. \( \nabla \cdot D = 0 \quad \nabla \times E = - \frac{\partial B}{\partial t} \) have the same forms as \( \nabla \cdot B = 0 \quad \nabla \times B = \mu_0 \frac{\partial E}{\partial t} \). The only difference is in coefficients.

The equipotential lines [Wangsness [106, p.77, (5-38)]] for \( \phi \) in Wangsness [106, p.78, Figure 5-8] are the
same as those [Wangness [106, p.257, (16-37)] for $A_z$ in Wangness [106, p.258, Figure 16-4]. However, $\eta = 2\pi \epsilon_0 \phi / \lambda$ in Wangness [106, p.77, (5-38)] and $\eta = 2\pi A_z / \mu_0$ in Wangness [106, p.257, (16-37)]. Thus, the electric field in Wangness [106, p.77, Figure 5-7; p.78, Figure 5-8] is perpendicular to the B-field in Wangness [106, p.258, Figure 16-4].

III. By Wangness [106, p.362, (21-77)],

$E' = C [ E \cos \alpha + (\mu \epsilon)^{-1/2} B \sin \alpha ]$ and $B' = C [ - (\mu \epsilon)^{1/2} E \sin \alpha + B \cos \alpha ]$.

Let $(E, B)$ represent the fields for electric dipole radiation and $(E', B')$ represent the fields for magnetic dipole radiation. Then

$\alpha = -90^\circ, C = 1$ [Compare Wangness [106, p.480, Figure 28-2] with Wangness [106, p.484, Figure 28-4]].

By Wangness [106, p.362, (21-77)],

$E' = -cB$ and $B' = E/c$. Then

$B \rightarrow E'/c, E \rightarrow cB'$, as shown in Wangness [106, p.482, (28-77)].

IV. $B' = (\mu_0 \epsilon_0 \lambda / \lambda) \hat{z} \times E$ [Wangness [106, p.357, l.1–l.6]].

Consequently, $\alpha = 90^\circ$ [Compare Wangness [106, p.77, Figure 5-7; p.78, Figure 5-8] with Wangness [106, p.258, Figure 16-4]].

Thus, $B' = (\mu_0 \epsilon_0 / \lambda) \hat{z} \times E$ is compatible with the case $\alpha = 90^\circ$ in Wangness [106, p.362, (21-77)].

Remark. The notations rather than ideas given in Wangness [106] are confusing.

Example 6.72. (How we tailor calculations to our needs)

We often do a lot of unnecessary calculations for the radiation zone: Sadiku [89, p.591, l.1–l.1–l.1; p.595, l.1–l.4; p.599, l.1–l.6; p.600, l.1–l.8] and Wangness [106, p.477, l.1–l.5–p.478, l.1; p.482, l.1–l.2–p.483, l.4]. However, most of them will never be used. p.734, l.1–l.1–l.2 in http://www.ece.rutgers.edu/~orfanidi/ewa/ch15.pdf shows how we should tailor calculations to our needs by avoiding unnecessary ones. The unnecessary calculations not only waste time and space but may also easily leave a gap in the theory due to the failure to provide the calculations that we should. For example, Sadiku [89, p.599, (13.33)] and p.792, (17.8.1) in http://www.ece.rutgers.edu/~orfanidi/ewa/ch15.pdf are given without proofs. We may use p.3, l.4–p.8, l.4 in http://www.ece.mcmaster.ca/faculty/nikolova/antenna_dload/current_lectures/L12_Loop.pdf to prove Sadiku [89, p.600, (13.35a) & (13.35b)] and p.792, (17.8.1) in http://www.ece.rutgers.edu/~orfanidi/ewa/ch17.pdf, p.6, (12.23) in http://www.ece.mcmaster.ca/faculty/nikolova/antenna_dload/current_lectures/L12_Loop.pdf can be proved by using p.792, (17.8.1) in http://www.ece.rutgers.edu/~orfanidi/ewa/ch17.pdf and Maxwell equations: $H = \nabla \times A$ and $\nabla \times H = \frac{\partial D}{\partial t} = j \omega \epsilon E$ [Wangness [106, p.375, (24-4)]].

Remark 1. Since $dI$[p.4, (12.14)] in http://www.ece.mcmaster.ca/faculty/nikolova/antenna_dload/current_lectures/L12_Loop.pdf is on xy-plane, A cannot have $\hat{r}$ or $\hat{\theta}$ components. Consequently, $A = A_\phi \hat{\phi}$ [p.5, l.5 in http://www.ece.mcmaster.ca/faculty/nikolova/antenna_dload/current_lectures/L12_Loop.pdf].

Remark 2. $I^\infty_0 \cos(n\phi) e^{j\cos\phi} d\phi = \pi^j J_\nu(z)$ [p.5, (12.19) in http://www.ece.mcmaster.ca/faculty/nikolova/antenna_dload/current_lectures/L12_Loop.pdf].

Proof. $\int_\pi^\pi e^{j\sin\theta} e^{-j\phi} d\theta = 2\pi J_\nu(z)$ [p.14, l.3 in http://www.math.psu.edu/papikian/Kreh.pdf].

Let $\phi = \pi / 2 - \theta$. Then
\[ \int_{-\pi}^{\pi} e^{j\cos \varphi} [\cos(n\varphi) + j\sin(n\varphi)] d\varphi = 2\pi j^n I_n(z). \]

However, \( \int_{-\pi}^{\pi} e^{j\cos \varphi} \sin(n\varphi) d\varphi = 0 \) (since the integrand is an odd function of \( \varphi \)).

**Example 6.73.** (A method had better emphasize its key ideas rather than the general outlook of the final result)

A method had better emphasize its essential ideas rather than the general outlook of the final result. By doing so, the description of the method will follow the natural thought flow: the cause first, the effect next. If we emphasize the general outlook of the final result, then the description of method will go against the natural thought flow. The latter approach chooses the hard way; one can hardly see the insight from it.

Example I. The key idea of Born–Wolf [13, p.102, l.7] is as follows:

If we emphasize the general outlook of the final result, then the description of method will go against the natural thought flow because the cart is put before the horse. Actually, one can easily prove all the results in II based on the process of presenting the key idea in I.

**Example 6.74.** (Only through a language tool that is accurate enough may a delicate statement be described)

The proof of

\[ \lim_{\delta x \to 0} \frac{1}{\delta x} \left( \int_{\sigma} F d\nu' - \int_{\sigma} F d\nu'' \right) = -\int_{\sigma} F \rho_x dS' \]

is confusing in both notation and language. The language tool that the authors use is neither clear nor accurate enough to describe such a delicate result. The following proof attempts to clarify the details:

**Proof.** I. Let \( B(P,a) = \{(x,y,z) | |(x,y,z) - P| < a\} \). Then \( \partial B(P,a) = \sigma \).

Let \( B(T,a) = \{(x,y,z) | |(x,y,z) - T| < a\} \). Then \( \partial B(T,a) = \sigma' \).

Let \( \delta S' \) be the surface element on \( \partial [(B(T,a) \setminus B(P,a)) \cup (B(P,a) \setminus B(T,a))] \cap \sigma \) pointing away from the volume \( (B(T,a) \setminus B(P,a)) \cup (B(P,a) \setminus B(T,a)) \).

Let \( \rho_x \) be the x-component of the unit radial vector \( \vec{\rho} \) pointing from \( P \) to \( \delta S' \).

Let \( \{A,B\} = \{\tilde{B}(T,a) \setminus \tilde{B}(P,a)\} \cap \{\tilde{B}(P,a) \setminus \tilde{B}(T,a)\} \cap \{(x,y,z) | y = 0\} \) [Assume \( P \) is the origin], where \( A \) is the upper point and \( B \) is the lower point [see Born–Wolf [13] p.899, Fig. 9)]. Then \( \angle APT = \angle TPB = 90^\circ \) (since \( \delta x \) is small). Thus, \( \delta S' \times \rho_x \) is the signed projection area of \( \delta S' \) onto the \( yz \)-plane.

II. For the right shaded region \( \tilde{B}(T,a) \setminus \tilde{B}(P,a) \), \( \vec{\rho} \) and \( \delta S' \) [by convention, \( \delta S' \) is the outward normal pointing away from the volume \( \tilde{B}(T,a) \setminus \tilde{B}(P,a) \) are antiparallel, so \( -\rho_x \times \delta S' > 0 \)]. Hence, \( d\nu' = -\rho_x \times \delta x \times \delta S' \) (because \( d\nu' > 0 \)).

III. For the left shaded region \( \tilde{B}(P,a) \setminus \tilde{B}(T,a) \), \( \vec{\rho} \) and \( \delta S' \) are parallel, so \( \rho_x \times \delta S' > 0 \). Hence, \( d\nu' = \rho_x \times \delta x \times \delta S' \), i.e., \( -d\nu' = -\rho_x \times \delta x \times \delta S' \).

IV. \( \int_{\Sigma} F d\nu' - \int_{\Sigma} F d\nu'' = \int_{B(T,a) \setminus B(P,a)} F d\nu' - \int_{B(P,a) \setminus B(T,a)} F d\nu' \) [Born–Wolf [13] p.899, Fig. 9]

\[ \to -\delta x \int_{\sigma} F \rho_x dS' \]

[By II and III].
Example 6.75. (Marcoscopic versus microscopic viewpoints)

Snell’s law states that the time required for the incident wavefront passes through \( AD \) equals the time required for the transmitted wavefront passes through \( AD \) [Hecht [50] p.100, Figure 4.19; Born–Wolf [13] p.38, (1)]]. In Wangsness [106 chap. 25], Wangsness [106] p.408, (25-18; p.416, (25-49)] are derived from marcoscopic Maxwell’s equations, especially on the boundary conditions [Wangsness [106] p.406, l.12]]. Thus, Maxwell’s equations is the common root of Snell’s law and Fresnel formulas. These two theorems can also be derived from Ewald–Oseen extinction theorem [Born–Wolf [13, p.111, (34)]]. Based on atomic theory, we can have a deeper understanding about the transmitted waves. For example, Ewald–Oseen theorems can also be derived from Ewald–Oseen extinction theorem [Born–Wolf [13, p.108, (23)]]]. Based on macroscopic Maxwell’s equations, especially on the boundary conditions [Wangsness [106, chap. 25]]. In Wangsness [106, p.408, (25-18; p.416, (25-49)] are derived from

Example 6.76. (\( \delta \) is a linear functional)

\[
\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \Delta \mathbf{D} = 0 \quad \text{[Born–Wolf [13] p.902, (10)].}
\]

\[ \mathbf{E} \mathbf{E} \]

\[ \text{Proof.} \quad \delta \text{ is a linear functional. That the authors fail to point out this important concept hidden behind this proof makes one doubt if they master distribution theory.} \]

\[ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \mathbf{D} = 0 \quad \text{[Born–Wolf [13] p.1, (1)].} \]

\[ \nabla \times \mathbf{H} = U(\mathbf{F})\nabla \times \mathbf{H}^{(1)} + U(\mathbf{F})\nabla \times \mathbf{H}^{(2)} + \delta(\nabla \mathbf{F}) \times (\nabla \mathbf{H}) \quad \text{[Born–Wolf [13] p.902, (6)].} \]

\[ \frac{\partial \mathbf{D}}{\partial t} = U(-\mathbf{F}) \frac{\partial \mathbf{E}}{\partial t}^{(1)} + U(\mathbf{F}) \frac{\partial \mathbf{E}}{\partial t}^{(2)} + \delta(\nabla \mathbf{F}) \frac{\mathbf{E}}{\partial D}. \quad \text{Hence,} \]

\[ \delta(\nabla \mathbf{F}) \times (\nabla \mathbf{H}) = \delta(\nabla \mathbf{F}) \frac{\mathbf{E}}{\partial D} \mathbf{D}. \quad \text{Therefore,} \]

\[ (\nabla \mathbf{F}) \times (\nabla \mathbf{H}) = \frac{\partial \mathbf{E}}{\partial D} \mathbf{D} \quad \text{(since} \quad \delta \text{ is a linear functional [Rudin [87] p.141, l.7])}. \]

\[ \square \]

Example 6.77. (A scientific textbook should be written with the audience in mind)

A scientific textbook should be written with the audience in mind: it should contain not only results, but also the method to obtain them. If the proof is long, we should divide it into several steps so that readers may check the work step by step.

\[ \mathbf{K}(\mathbf{e}, \mathbf{n}) + \frac{1}{ik_0} \mathbf{L}(\mathbf{e}, \mathbf{n}, \mu) + \frac{1}{ik_0^2} \mathbf{M}(\mathbf{e}, \mu) = 0 \quad \text{[Born–Wolf [13] p.119, (16)].} \]

\[ \mathbf{K} \]

\[ \mathbf{M} \]

\[ \mathbf{L} \]

\[ \mathbf{E} \]

\[ \text{Proof.} \quad \nabla \mathbf{E} = \frac{ik_0}{c} \{ \nabla \mathbf{e} + 2ik_0 \nabla \mathbf{S} \cdot \mathbf{e} + ik_0 \mathbf{e} \cdot \nabla \mathbf{S} + (ik_0^2) (\nabla \mathbf{S})^2 \mathbf{e} \}. \]

\[ \text{Proof.} \quad \nabla \mathbf{E} = \frac{ik_0}{c} \{ \nabla \mathbf{e} + 2ik_0 \nabla \mathbf{S} \cdot \mathbf{e} + ik_0 \mathbf{e} \cdot \nabla \mathbf{S} + (ik_0^2) (\nabla \mathbf{S})^2 \mathbf{e} \}. \]

\[ \nabla (\ln \mu) \times (\nabla \times \mathbf{E}) = \nabla (\ln \mu) \times (\nabla \times \mathbf{e} + i k_0 (\nabla \mathbf{S}) \times \mathbf{e}) e^{ik_0 S} \quad \text{[Born–Wolf [13] p.118, (9)]} \]

\[ = \{ \nabla (\ln \mu) \times (\nabla \times \mathbf{e}) + ik_0(\nabla \mathbf{S}) (\nabla (\ln \mu) \cdot \mathbf{e}) - \mathbf{e} (\nabla (\ln \mu) \cdot (\nabla \mathbf{S})) \} e^{ik_0 S} \quad \text{[Wangsness [106] p.11, (1-30)].} \]

\[ \nabla (\mathbf{E} \cdot \nabla (\ln \mu)) = \{ \nabla (\mathbf{e} \cdot \nabla (\ln \mu)) \} e^{ik_0 S} + (\mathbf{e} \cdot \nabla (\ln \mu)) (ik_0 e^{ik_0 S} \nabla \mathbf{S}). \]

\[ \square \]

Example 6.78. (Tracing to the orientation’s root from which all its derivatives come)

1. For any mathematical concept, we should trace it to its origin so that we may understand it more deeply. For example, the definition of curvature given in Weatherburn [110] vol. 1, p.11, l.7-1 is natural, original, and universal, while that given in O’neill [74] p.57, l.2] is more artificial, consequential, and less universal. The root (prototype) of orientation is positive unit circle \( (\cos \theta, \sin \theta), \theta \in [0, 2\pi] \) or the right-handed coordinate system. All the following concepts concerning orientations can be derived from this prototype: Curve orientation: [https://en.wikipedia.org/wiki/Curve_orientation], Courant–John [25] vol. 2, p.86, l.22–l.25; p.587, l.17+l.15]


Oriented area within a closed curve: Courant–John [25] vol. 1, p.365, l.15–l.18]
Right-handed screws: Courant–John \cite{25} vol. 2, p.185, l.–5–l.–1]
Orientation of the triple \((A,B,A \times b)\) = Orientation of \((E_1,E_2,E_3)\) [Courant–John \cite{25} vol. 2, p.185, l.–14–l.–12]\]
Orientation of \(\mathbb{R}^n\): Carmo \cite{18} p.12, l.11\]
Orientation of a manifold: Spivak \cite{95} vol.1, p.118, l.4
Positive curvature: Weatherburn \cite{110} vol. 1, p.12, l.–3
Positive torsion: Weatherburn \cite{110} vol. 1, p.15, l.1
Positive geodesic curvature: Kreyszig \cite{61} p.137, l.–5; Weatherburn \cite{110} vol. 1, p.109, l.12
Remark: Spivak \cite{95} is burdened with manifolds and Kreyszig \cite{61} is burdened with tensors. Manifolds are the generation of Euclidean space and tensors are convenient for coordinate changes, but they are not the essential kernel of differential geometry. In contrast, O’Neill \cite{74} and Weatherburn \cite{110} are concrete and intuitive, so they are good for practical usage. Note that the unrigorous geometric proof given in Weatherburn \cite{110} vol. 1, p.42, l.3–l.11 can be rigorously proved by calculus \cite{as Weatherburn \cite{110} vol. 1, p.12, l.6–l.8} and is obviously simpler than the analytic proof given in Weatherburn \cite{110} vol. 1, p.42, l.12–p.43, l.12.

II. The point \(r_1\) lies in the normal plane to the given curve [Weatherburn \cite{110} vol. 1, p.32, l.13–l.14].

Proof. \([r_1(s), r(s)]\) lies on the tangent line at \(r_1(s)\) which is perpendicular to \(t(s)\) [Kreyszig \cite{61} p.52, l.18]. Since the normal plane at \(r(s)\) is perpendicular to \(t(s)\) [Kreyszig \cite{61} p.32, Fig. 10.2], \([r_1(s), r(s)]\) lies on the normal plane at \(r(s)\).

Remark. The definition of involute given in Kreyszig \cite{61} p.52, l.18 is original. \url{http://mathworld.wolfram.com/Involute.html} uses the consequential property [Kreyszig \cite{61}, (15.2); l.–4–l.–1] of the above definition as the definition. The former definition is a simple characteristic property, while the latter definition provides one procedure of construction. When considering the converse problem [Kreyszig \cite{61} p.53, l.–4–l.–1], we would like to choose the former definition because there are fewer steps required to be reversed. See the above proof. If we use the former definition, there is a single infinitude of evolutes [Weatherburn \cite{110} vol. 1, p.33, l.–1–p.34, l.1]. If we use the latter definition, the curve has a unique evolute [\url{http://mathworld.wolfram.com/Involute.html}]. This is because the horizontal line segment given in Weatherburn \cite{110} vol. 1, p.30, Fig. 6] must be in the direction of the principal normal at \(r_1\) [\(c\) in Kreyszig \cite{61} p.54, (15.6)] must be \(\frac{\pi}{2}\). Thus, the latter definitions destories the symmetry between involutes and evolutes.

Example 6.79. (Reading classics in modern times)
I. The proof of Weatherburn \cite{110} vol. 1, p.66, (2)] requires the following lemma:
The normals \(n, n + dn\) of consecutive points intersect if and only if \([n, n + dn, dr] = 0\) [Weatherburn \cite{110} vol. 1, p.66, l.11–l.12]. There are two problems in the proof of this lemma:
1st problem: \(n, n + dn\) can be parallel and coplanar. \(n, n + dn\) fail to intersect in this case.
2nd problem: To construct normals at consecutive points involves two limiting processes: the construction of normals is one and the construction of a circle tangent to the curve is another one. Which one should be done first? If we exchange their order, can the results be different? The argument given in Weatherburn \cite{110} vol. 1, p.66, l.4–l.15] fail to answer any of these questions. Therefore, if we want to preserve the original idea, we should complete one of the limiting processes first.
II. Fortunately, Kreyszig \cite{61} p.91, (27.4)] and Weatherburn \cite{110} vol. 1, p.66, (2)] are the same, so we can make the proof of the latter rigorous by referring to the proof of the former. The proof of Kreyszig \cite{61} p.91, (27.4)] is given in Kreyszig \cite{61} p.90, l.–13–p.91, l.2]. It contains two steps: 1. Find the tensor formula
II. Having rigorously proved Weatherburn [110, vol. 1, p.66, (2)] by the tensor approach, we would like to directly fill the gap in the lemma in I. The procedure to find the maximum curvature can be divided into two steps: 1. Choose a normal section, construct its osculating circle, and find the normal curvature $\kappa_n$. Note that the normal is the principal normal [Weatherburn [110, vol. 1, p.61, l.−3–l.−2]]. 2. By Euler’s formula [O’neill [74, p.201, Corollary 2.6]], we may find the maximum curvature.

Suppose we have finished step 1 and replace the curve in the normal section with the osculating circle of radius $R$ [Kreyszig [61, p.82, l.−2]] at $P$. Then the principal normals at consecutive points intersect if and only if the two principal normals are coplanar because they must intersect at the center of the circle. Once the lemma in I is proved, the rest of proof just attempts to make $\vartheta$ in O’neill [74, p.201, Corollary 2.6] approach zero.

IV. Kreyszig [61] can be considered a bridge between the classic textbook Weatherburn [110] and the modern textbook O’neill [74]. Classic textbook pays too much attention to the computations on matrix elements $[a_{ij}$ in Carmo [18, p.154, l.14]] of an operator, while the modern textbook tries to attach clear geometric meanings to the operator $[dN$ in Carmo [18, p.154, l.14]]. For example, the definition given in O’neill [74, p.196, Definition 2.2] is compatible with the second formula of O’neill [74, p.58, Theorem 3.2]. Note that Kreyszig [61, p.83, Theorem 24.2] is not as clear as Weatherburn [110, vol. 1, p.62, (22)].

Kreyszig [61, p.90, l.−13–p.91, l.2] proves the following theorem:

If $P$ is not umbilic, then (the direction of $\frac{d^2}{da^2}$ at $P$ which has an extreme curvature) if and only if (Kreyszig [61, p.91, (27.4)] holds).

If we compare the proof of this theorem with that of O’neill [74, p.200, Theorem 2.5(2)], we find that the latter approach greatly reduces awkward computations by taking advantage of eigenvectors of the shape operator. For example, the matrix of the shape operator becomes simple (see O’neill [74, p.213, Lemma 4.2]), while its general matrix elements are complicated [Kreyszig [61, p.80, l.3; (23.9)]].

Furthermore, the fact that the principal directions are orthogonal [O’neill [74, p.200, l.12]] can be easily seen by a famous theorem (two eigenvectors corresponding to different eigenvalues are orthogonal) in linear algebra, while the proof given in Kreyszig [61, p.92, l.−7–l.−5] is complicated.

V. A. Weatherburn [110, vol. 1, p.22, l.5–l.7] provides the key to translating the classic language into the modern language.

Examples.

The osculating circle has two consecutive tangents (three consecutive points) in common with the curve [Weatherburn [110, vol. 1, p.13, l.12–l.13]] → The osculating circle has contact of second order with the curve [Kreyszig [61, p.51, l.−10–l.−9]].

The osculating sphere has four consecutive points in common with the curve [Weatherburn [110, vol. 1, p.22, l.5]] → The osculating sphere has contact of third order with the curve [Kreyszig [61, p.51, l.−6]].

B. How Struik [98] improves classical differential geometry and how formalism invades modern differential geometry

Expressions using consecutive points, consecutive tangents, consecutive osculating planes [Struik [98, p.12, Fig. 1-13]] have considerable heuristic value and can still be made quite rigorous [Struik [98, p.7, l.14–l.16]].

(1). Osculating plane

(i). The equation of osculating plane $(X - x, \dot{x}, \ddot{x}) = 0$ [Struik [98, p.12, (3-10)]] is established based on Rolle’s theorem [Struik [98, p.10, l.−9]], while $[R - r, \dot{r}, \ddot{r}] = 0$ [Weatherburn [110, vol. 1, p.12, l.11]] is established based on intuition [Weatherburn [110, vol. 1, p.12, l.5]]. The definition given in Kreyszig [61]
p.33, Table 10-1 is artificial and formal rather than natural and original. Formalism may easily make an argument rigorous, but may lose the natural and original taste. It is convenient for mechanic application because we need not consider origins.

(ii). $n \perp t$ [Weatherburn [110] vol. 1, p.11, l.−1] should have been proved by Struik [98 p.13, (4.2)].

(iii). Struik [98 p.10, l.14–p.14, l.3] shows that a plane passing through three consecutive points of a curve [osculating plane] has the form given in Struik [98 p.11, (3.4a)].

(2). Curvature
The definition of curvature $\kappa$ in Weatherburn [110] vol. 1, p.11, l.–8] is based on intuition. The same formula [Struik [98 p.14, (4-5)] used for definition is proved by using Struik [98 p.13, (4-3)(ii); p.14, l.9–l.10].

(3). Osculating circle
(i). The definition of the circle of curvature given in Weatherburn [110] vol. 1, p.13, l.8–l.9] is based on intuition [a circle that passes through three consecutive points of a curve]. The definition given in O’Neill [74] pp. 64–65, Exercise 6] uses the concept of contact of order two instead of three consecutive points. Struik [98 p.14, l.16–p.15, l.7] provides the solution of O’Neill [74 pp. 64–65, Exercise 6]. The concept of contact of finite order belongs to formalism. It hides the true geometric meaning [Struik [98 p.23, l.16–l.22] → Kreyzig [61, p.48, l.4–l.11]]. It also eliminates the origins [Rolle’s theorem] by replacing the formulas given in Struik [98 p.10, l.−9–l.−6] with Struik [98 p.10, l.33; p.14, (4.7)].

(ii). The osculating circle lies in the osculating plane at $P$ [Struik [98 p.14, l.18–l.19]]. Classical proof using consecutive points is based on intuition.
Modern proof: The osculating circle passes through three consecutive points of the curve and lies in a plane. Struik [98 p.10, (3-3)] shows that a plane contains three consecutive points of the curve [osculating plane] is a plane that has contact of order two with the curve. The latter plane is generated by $x', x''$ [Struik [98 p.11, (3-4a)]]. This property is used in Struik [98 p.14, l.–1].

VI. The drawbacks of classical language: A concise definition cannot be not easily isolated from a long context; the background information is unclear and confusing; proofs are not rigorous.

1. The definition of principal surfaces for a ray is given in Weatherburn [110] vol. 1, p.13, l.8–l.9]. How we abstract a concise definition of principal surfaces from Weatherburn [110] vol. 1, p.184, l.–5–p.185, l.–6] seems to be a difficult problem.

2. What is the background information about the point of contact given in Weatherburn [110] vol. 1, p.141, l.–9]? Let $r(u) = r_0 + ud$ be the generator of a ruled surface [Weatherburn [110] vol. 1, p.139, l.15]]. The point of contact refers to the point $r(u)$ whose tangent plane to the ruled surface may be different from the tangent planes at other points of the generator. The purpose of the statement given in Weatherburn [110] vol. 1, p.141, l.–11–l.–8] is to label the tangent planes through the generator [Weatherburn [110] vol. 1, p.140, l.–2–l.–1; p.141, (30)]. Any tangent plane through the generator is labeled by the point at which the plane is tangent to the ruled surface.

3. What is the background information about the normal plane given in Weatherburn [110] vol. 1, p.141, l.–4]? A normal plane must belong to a curve; the curve is the generator. A normal plane must have a principal normal; the principal normal lies on the tangent plane to the ruled surface at $r(u)$. It is the principal normal that makes the tangent plane turn [Weatherburn [110] vol. 1, p.141, l.10] because the tangent vector is fixed.

4. The definition of principal planes is based on limits, but the role that limits play in defining principal planes is ignored in the argument given in Weatherburn [110] vol. 1, p.186, l.–4–l.–1].

VII. Of all sections through two consecutive points $P, Q$ on the surface, the normal section makes the length of arc $PQ$ a minimum [Weatherburn [110] vol. 1, p.99, l.16–l.18]]. The arc length $PQ$ requires a careful explanation.
Proof. Draw a circle through points $P, Q$. The larger the radius is [the smaller the curvature is], the smaller the arc length $PQ$ is.

VIII. Modern mathematics lacks depth and completeness. Weatherburn [110] vol. 1, p.90, (1) provides more information than O’neill [74] p.213, Lemma 4.2. Thus, the information that the latter provides is incomplete. When considering a deep problem like Weatherburn [110] vol. 1, p.100, (6), the latter would fail to provide an adequate tool to solve the problem.

IX. (Actions speak louder than words: improvement by applying differentiation to linear algebra [Blaga [11] §1.14.1])

The following improvement shows that proving an idea step by step takes some effort:
X. In modern differential geometry, the form of differential equation for curves has a closer relationship to the parametric representation of these curves.
Principal curves: Kreyszig [61] p.91, (27.4) → O’neill [74] p.230, Exercise 6(a)).

XI. Modern differential geometry is more organized and reveals more insights and geometric meanings.

If we compare the proof given in O’neill [74] p.220, Exercise 9] with the proof given in Weatherburn [110] vol. 1, p.72, l.–21–l.–5], the former proof shows the role that the condition $m = 0$ plays in principal curves, and expresses the principal curvatures as $l/E, n/G$, while it is difficult to recognizes these features in the latter proof.

XII. (The modern version is not as clear, heuristic, complete, and organized as the original version and may easily emphasize on the trivial part) For the existence part of the fundamental theorem of curve theory, the proof of Eisenhart [31] p.24, l.–6–l.–1] is clearer, more heuristic and organized than those of Struik [98] p.29, l.–10–p.31, l.8] and Blaga [11] p.72, Theorem 1.14.3]. The transformation from the existence part of the fundamental theorem of curve theory to Riccati equations is direct and smooth in Eisenhart [31] p.25, l.1–l.16], while the transformation gets interrupted in both Struik [98, §1-10] and Blaga [11, §3.1]. The lack of summary given in Eisenhart [31] p.25, l.13–l.16] makes the discussion in both Struik [98, p.36, l.1–l.–1] and Blaga [11, §3.1] look incomplete and disorganized.

Example 6.80. (How we name a definition)
I. The definition of conjugate directions [Weatherburn [110] vol. 1, p.80, l.6–l.7]) is introduced at the beginning of Weatherburn [110] vol. 1, §35], but we know neither its prototype nor from where its name comes until we read up to the end of that section [Weatherburn [110] vol. 1, p.81, l.–11–l.–10]]. The name comes from the consideration of the picture given in Bell [6] p.115, Fig. 36]. The definition is based on its analytic property useful for proofs.
II. The definition of asymptotic directions is given in Weatherburn [110] vol. 1, p.83, l.2–l.3]. The name comes from the asymptotes of the indicatrix [Weatherburn [110] vol. 1, p.83, l.17–l.18]], i.e., from the consideration of the picture given in Kreyszig [61] p.85, Fig. 25.3]. The definition is also characterized by its analytic property useful for proofs. We could give the definition a more direct name such as self-conjugate directions, but for the mathematicians in the ninefteenth century the original picture was their first choice for naming a definition.

Example 6.81. (Advantageous viewpoints)
When studying mathematics, we should take an advantageous viewpoint to get to the heart of the matter in few words. Weatherburn [110] vol. 1, p.106, l.–15–l.–7] says a lot, but fails to hit the heart of the matter. In fact, the passage is difficult to read, but we may easily understand its key idea by taking the following
viewpoint: If any two terms in Weatherburn [110, vol. 1, p.105, (14)] vanish, so does the third term. By Weatherburn [110, vol. 1, p.105, Fig. 14], the angle between the normal of the surface and the normal of the plane is $\frac{\pi}{2} - \sigma$.

**Example 6.82.** (We should look for the clue to a solution in concrete examples; natural order of thought flow)

I. Theory is the framework of mathematics, while examples are the flesh of mathematics. When we try to solve a theoretical problem, we should resort to concrete examples for clues. I had a difficulty in understanding the last sentence in Kreyszig [61, p.49, l.9–l.15]. By a proper rotation, we may assume that $(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_3})|_{s_0} = (0, 0, 1)$, but it is difficult to see how we make $\frac{d^{m+1} \alpha_1}{ds^{m+1}} \neq \frac{d^{m+1} \beta_1}{ds^{m+1}}$ to ensure $p^{(m+1)}(s_0) \neq 0$ [Kreyszig [61, p.49, l.18]]. The difficulty lies in how we exclude other possible options: $\frac{d^{m+1} \alpha_1}{ds^{m+1}} \neq \frac{d^{m+1} \beta_1}{ds^{m+1}}$ or $\frac{d^{m+1} \alpha_0}{ds^{m+1}} \neq \frac{d^{m+1} \beta_0}{ds^{m+1}}$. Consequently, I look for clues in the concrete examples given in Blaga [11] §1.12–§1.13. I have found that the part (a) of the proof of Kreyszig [61, pp. 48–49, Lemma 14.2] lacks the following information:

1. “This yields (14.3)” [Kreyszig [61, p.49, l.9–l.15]] should have been replaced with “By the argument in (b), this yields (14.3)” because we have assigned $\beta_1 = \alpha_1(s), \beta_2 = \alpha_2(s)$ there. This assignment eliminates the above difficulty.

2. The coordinate frame given in Blaga [11, p.63, l.9–l.15] is the key to understanding the geometric meaning of coordinate axes and this fact should not be omitted because the coordinate representation of the surface $G$ in Kreyszig [61, pp. 48–49, Lemma 14.2] has to be consistent with this coordinate frame. Note that $\{M_0; \tau_0, \nu_0, \beta_0\}$ is a fixed coordinate frame; coordinate changes will not change the shape of the curve. A curve represented in this coordinate frame preserves more geometric information, simplifies the computation of its order of contact with a special type of surfaces, and provides the proof of Kreyszig [61, pp. 48–49, Lemma 14.2] with a specific and solid explanation.

Remark. The computation given in Blaga [11, §1.13] is neater and more straightforward than that given in the proof of Kreyszig [61, p.51, Theorem 14.4]. Let us use the strategy given in Blaga [11, §1.13] to prove the statement given in Kreyszig [61, p.51, 1.5]. The proof also shows how we should rigorously prove the statements given in Weatherburn [110, vol. 1, p.13, 1.8–1.18].

Proof. Let $F(x, y, z) = x^2 + y^2 - 2ax - 2by = 0$.

\[
\begin{align*}
[s - \frac{1}{6}k^2(0)s^3 + o(s^3)]^2 + [\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)]^2 - 2a[s - \frac{1}{6}k^2(0)s^3 + o(s^3)] - 2b[\frac{1}{2}k(0)s^2 + \frac{1}{6}k'(0)s^3 + o(s^3)] &= 0 \quad \text{[Blaga [11], p.64, (1.11.9)]}, \\
-2as + [1 - bk(0)]s^2 + [\frac{1}{2}k^2(0) - \frac{1}{6}k'(0)]s^3 + o(s^3) &= 0.
\end{align*}
\]

Let $a = 0$ and $1 - bk(0) = 0$. Then the circle with center $(0, \frac{1}{k(0)}, 0)$ has contact of second order (at least) with $C$ at $P = (0, 0, 0)$.

II. (Natural order of thought flow) From the hindsight, we see that the following definition is more natural and original than the definition given in Kreyszig [61, p.48, Definition 12.2]:

Let $G(\alpha_1(s), \alpha_2(s), \alpha_3(s)) = \sum_{n=1}^{\infty} a_n s^n$. The surface $G(x_1, x_2, x_3) = 0$ has contact of order $m$ with the curve $\alpha$ if $a_i = 0 (i = 0, \cdots, m)$ and $a_{m+1} \neq 0$.

In fact, this is the only natural way to define contact of order $m$. Thus, we should have used this new definition and Kreyszig [61, pp. 48–49, Lemma 14.2] to prove Kreyszig [61, p.48, Definition 12.2] as a theorem.

**Example 6.83.** (Reasons that lead to confusion or difficulty)

Some statements in Kreyszig [61 §24] are confusing or difficult to understand. Let us analyze the
Reasons that lead to confusion or difficulty, and then find ways to improve them.

1. Failure to manage complexity with simplicity.

Kreyszig [61, p.82, Theorem 24.1] follows from the following lemma and the statement given in Kreyszig [61, p.82, l.6–l.7]:

Lemma. Curves on the surface that have the same osculating plane at \( P \) will have the same tangent vector at \( P \).

Proof. The normal of the surface at \( P \) lies on the normal plane of each of these curves [Kreyszig [61, p.78, l.1–l.7]].

The curves with the same osculating plane have the same binormal.

The common binormal of these curves and the normal of the surface determine the common normal plane, and \( \gamma \) determines the common principal normal. The desired result follows from Weatherburn [110, vol. 1, p.13, l.1–l.3].

Remark 1. How do we interpret the \( du^1 : du^2 \) given in Kreyszig [61, p.81, (24.1)]? For the curves in Kreyszig [61, p.82, Theorem 24.1], Weatherburn [110, vol. 1, p.62, l.8–l.7] shows that \( du^1 : du^2 \) for the general case is the same as \( du^1 : du^2 \) for the case of the normal section. These curves are distinguished by \( \gamma \) which is on the left-hand side of Kreyszig [61, p.81, (24.1)].

Remark 2. Kreyszig [61, p.82, Theorem 24.1; p.83, Theorem 24.2] look like two complicated theorems. In fact, the former is a side issue of Weatherburn [110, vol. 1, p.62, (22)] and can be considered its corollary; the latter is just a variant of Weatherburn [110, vol. 1, p.62, (22)].

2. The intersection of \( S \) and \( O(P) \) is a plane curve [Kreyszig [61, p.82, l.13–l.14]].

Proof. \( O(P) \) is not the tangent plane at \( P \), so the intersection contains more than one point.

3. The statement fails to provide the details.

If this tangent is fixed, so is the right-hand side of (24.1) [Kreyszig [61, p.82, l.17–l.18]].

Proof. If the direction of tangent vector \( x'(s) = x_1(s) \frac{du^1(s)}{ds} + x_2(s) \frac{du^2(s)}{ds} \) is fixed, so is \( du^1(s) : du^2(s) \). Therefore, the right-hand side of (24.1) is fixed.

Remark. “This tangent” refers to the direction of the tangent vector instead of the tangent vector.

4. The statement fails to hit the heart of the matter.

The sign of \( \cos \gamma \) depends on the orientation of \( S \) and is reversed if the transformation of coordinates \( u^1, u^2 \) reverses the senses of the unit normal vector.

Proof. The numerator of the right-hand side of Kreyszig [61, p.81, (24.1)] comes from \( x_{jk} \cdot n \). Thus, if we change \( n \) to \( -n \), then the sign of \( \kappa_n \) will be reversed.

5. The statement fails to focus on the key idea or use right terms.

The key idea of Meusnier’s theorem [Weatherburn [110, vol. 1, p.62, l.3]] is as follows: We discuss the plane sections of surface \( S \) whose tangents at \( P \) have the same direction. They are plane curves. We choose a standard curve among them and try to figure out how it is related to other curves. Rotate the normal section by using the normal as the axis until its tangent at \( P \) has the assigned direction. Then this
normal section is regarded as the standard curve (One may use Kreyszig \[61\] p.83, Example 24.2\] as an example). The problem with Kreyszig \[61\] p.83, Theorem 24.2\] is that it fails to choose the standard curve for comparison, and thus blurs the key idea. The term “normal plane” used in Weatherburn \[110\] vol. 1, p.62, l.1–13\] may easily be misinterpreted. The term “hyperboloid” given in O’neill \[74\] p.204, l.14\] should have been corrected as “hyperbolic paraboloid”.

6. Failure to distinguish one standpoint from the other.

The matrix for Marion \[70\] p.9, l.16\] is different from the rotation matrix given in \url{https://en.wikipedia.org/wiki/Rotation_matrix}. The difference comes from different standpoints. The standpoint for the former matrix: \(P\) is fixed; the coordinate axes are rotated by angle \(\theta\); \((x_1, x_2)\) is the coordinates of \(P\) with respect to the original coordinate axes, while \((x'_1, x'_2)\) is the coordinates of \(P\) with respect to the rotated coordinate axes. The standpoint for the latter matrix: the coordinate axes are kept the same; \(P\) is rotated to \(P'\) by angle \(\theta\); \((x_1, x_2)\) is the coordinates of \(P\), while \((x'_1, x'_2)\) is the coordinates of \(P'\).

7. Failure to indicate the conditions under which a statement is true.

(a). \(EN - GL \neq 0\) [Weatherburn \[110\] vol. 1, p.72, l.13\].

(b). This determinant is zero only at umbilics [Kreyszig \[61\] p.93, l.12].

The proof of Kreyszig \[61\] p.93, Theorem 27.4\] should be divided into two parts: \(\Rightarrow\) and \(\Leftarrow\). Statement (b) is used only in part \(\Leftarrow\). The proof of Kreyszig \[61\] p.93, Theorem 27.4\] is incorrect because it misplaces statement (b) in part \(\Rightarrow\). In part \(\Rightarrow\), we use \(dudv = 0\) [Weatherburn \[110\] vol. 1, p.72, l.16\] to prove statement (a); in part \(\Leftarrow\), we should use statement (b) to prove statement (a). The proof given in Weatherburn \[110\] vol. 1, p.72, l.16–l.8\] contains a gap because it fails to use statement (b) to prove statement (a).

**Example 6.84.** (The prototype of a concept and the compatibility between it and its derivatives)

I. A definition of a mathematical concept contains a starting point (i.e., standpoint or viewpoint) and a special path leading to the concept. We should put the starting point at the prototype of the concept rather than its equivalent derivatives because of the advantage of viewing all the paths leading to the concept. For the concept of regularity, O’neill \[74\] p.38, Definition 7.9; l.1–10\] [Kreyszig \[61\] p.60, l.1–8\] is the prototype, while Kreyszig \[61\] p.56, l.1–10\] and Blaga \[11\] p.17, l.4; p.39, l.10; p.43, l.4\] are its derivatives. Whenever we read a derivative for the first time, we should check the compatibility between it and the prototype. If in a textbook the author puts the prototype after its derivatives, the readers will actually lose the opportunity of organizing the topic by checking the compatibility of a derivative with the prototype. See Blaga \[11\] p.115, l.1–10.

II. Covariant derivatives computed along the parameter curves of \(x\) reduce to [“reduce to” should have been corrected as “become”] partial differentiation with respect to \(u\) and \(v\) [O’neill \[74\] p.211, l.11–l.12].

**Explanation.** Based on the same idea, the special case \(\nabla_\alpha Z = (Z \circ \alpha)'(0)\) [O’neill \[74\] p.78, l.2; p.81, l.5\] can be generalized to the general case \(\nabla_\alpha W = W'(p + tv)'(0)\) [O’neill \[74\] p.78, l.2].

\[\nabla_{0,1}x = (x((u_0, v_0) + t[1, 0]))'(0) = \frac{\partial x}{\partial s}(u_0, v_0).\]

Remark. In the above explanation, the first equation is 1-dimensional, while the third equation involves the 2-dimensional surface. Considering the dimensions that the two equations involve, I think that “become” is more appropriate than “reduce to”.

**Example 6.85.** (The consistency of derivatives; the theory of differentiation in tensors runs parallel to that in differential forms; the consistency between the derivative of a map from one surface to another surface and the derivative of a map from one Euclidean space to another Euclidean space)

I. The consistency between the exterior derivative of 1-form in \(\mathbb{R}^3\) [O’neill \[74\] p.32, Exercise 8(b)] and
the exterior derivative of 1-form on a surface [O’Neill [74] p.154, Definition 4.4] can be seen through the exterior derivative of 1-form in \( \mathbb{E}^2 \) [O’Neill [74] p.156, l.4-5].

II. \( \nabla Z = (Z_\alpha)'(0) \) [O’Neill [74] p.190, l.1].

**Proof.**

**Proof.** \( \nabla (\text{Tensors}) \)

\begin{align*}
\nabla Z &= \sum_{\alpha} \nabla (a(\alpha)) \left( Z_i U_i(p) \right) [O’Neill [74] p.78, Lemma 5.2] \\
&= \sum_{\alpha} \frac{dz(a(\alpha))}{dt} U_i(p) [O’Neill [74] p.19, Lemma 4.6] \\
&= (Z_\alpha)'(t) [O’Neill [74] p.189, l.2]. \\
&= (Z_\alpha)'(t) \quad \square
\end{align*}


III. The expression for derivatives


**Remark.** This is not the complete story. The consummation along this direction is the identification of the two proofs of different versions of Stokes’ theorem: O’Neill [74] p.170, Theorem 6.5 and Wangsness [106] p.24, (1-67).

**Identification.** Let \( \phi = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 \) [O’Neill [74] p.24, Lemma 5.4].

\begin{align*}
\int_{S} d\phi &= \int_{S} \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 dx_2 + \left( \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) dx_1 dx_3 + \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) dx_2 dx_3 [O’Neill [74] p.29, l.3] \\
&= \int_{C} \nabla \times (f_1, f_2, f_3) \cdot da \quad \square \\
&= \int_{C} (f_1, f_2, f_3) \cdot ds \quad \square
\end{align*}

IV. Although \( F \) for surfaces is defined as in O’Neill [74] p.35, Definition 7.4, the version given in O’Neill [74] p.40, Exercise 8 is used most often. The latter version can be considered a special case of O’Neill [74] p.38, Theorem 7.8] when \( t = 0 \).


**Proof.** By O’Neill [74] p.144, Lemma 3.1], a curve \( \alpha \) in \( M \) can be expressed as \( \alpha(t) = x(a_1(t), a_2(t)) \), where \( a_1(t), a_2(t) \) are differentiable.

\( \alpha' = (F\alpha)' = (Fx(a_1, a_2))' = (y(a_1, a_2))' = a_1'(t)y_u(a_1, a_2) + a_2'(t)y_v(a_1, a_2) \).

\( \nabla \alpha' = \nabla (F\alpha)' = \nabla (Fx(a_1, a_2))' = \nabla (y(a_1, a_2))' = \nabla a_1'(t)y_u(a_1, a_2) + \nabla a_2'(t)y_v(a_1, a_2) \).

\( \square \)


**Proof.** A. The justification of \( F_\ast : T_p(M) \to T_{F(p)}(N) \)

In the neighborhood of \( p \) and the neighborhood of \( F(p) \), \( M \) and \( N \) can be considered planes, curves in the surfaces can be considered straight lines, and \( F(q) = F(p) \) can be considered \( F'(q - p) \) by the Taylor series approximation [Rudin [86] p.188, Definition 9.10].

B. The construction of \( F_\ast : T_p(M) \to T_{F(p)}(N) \)

By O’Neill [74] p.125, Definition 1.2], a neighborhood of \( p \) in \( M \) can be contained in a proper patch \( x(u, v) \) in \( M \). Let \( y = Fx. \)

\( (Fx)_{u} [1,0] = \frac{\partial (Fx)(u,v)}{\partial u} [O’Neill [74] p.38, Theorem 7.8] \)

\( = (Fx)_u = y_u(u,v). \)

Similarly, we define \( F_x(x_u) = (Fx)_u [1,0] = y_u. \)
Remark. O’Neill [74, p.165, l.23] provides a geometric meaning of through a reparameterization of the curve $\alpha$.

\[ \frac{dg}{dt} = \alpha', \]

C. $F\alpha' = (F\alpha)'$ for every curve $\alpha$ in $M$.

**Proof.** $F\alpha'(t) = F_\ast(\alpha'(t)x_u + \alpha'(t)x_v)$ [See O’Neill [74] p.166, Exercise 12]

\[ = a_1'(t)y_u + a_2'(t)y_v \quad [\text{by B}] \]

\[ = (F\alpha)'(t) \quad [\text{by O’Neill [74] p.166, Exercise 12}]. \]

Remark. For applications, we should understand the geometric meanings of $F_\ast\alpha' = (F\alpha)'$. First, $\alpha'(t)$ is the tangent vector of the curve $\alpha$ at $\alpha(t)$, so it is in $T_p(M)$, the domain of $F_\ast$ [O’Neill [74] p.161, l.12]. This chain rule for curves in a surface is the same as the chain rule for curves in an Euclidean space [O’Neill [74] p.38, Theorem 7.8] except that the domain of $F_\ast$ should be changed from an Euclidean space to a surface. This exercise shows how to remove the obstacle for the change. The key idea of O’Neill [74] p.19, Lemma 4.6; p.149, Definition 3.10] is the chain rule. Let us prove O’Neill [74] p.165, Exercise 8].

**Proof.** (a). $(F(\alpha')|g] = (F\alpha)'|g] \quad [\text{by O’Neill [74] p.166, Exercise 13}]

\[ = \frac{d(g\alpha)(t)}{dt} \quad [\text{We may identify O’Neill [74] p.149, Definition 3.10} \text{ with O’Neill [74] p.166, Exercise 13} \text{ through a reparameterization of the curve}] \]

\[ = \alpha'(t)[g(F)] \quad [\text{We may identify O’Neill [74] p.149, Definition 3.10} \text{ with O’Neill [74] p.166, Exercise 13} \text{ through a reparameterization of the curve}] \]

Remark. O’Neill [74] p.165, l.123] provides a geometric meaning of $v[g(F)] = (F,v)[g]$. I interpret $\alpha'(t)[g(F)] = (F,\alpha')(t)[g]$ as a consequence of the chain rule. This is because the chain rule has many forms, but their idea is the same. Actually, $\alpha'(t)[f]$ is a confusing notation; I would rather use $f, \alpha'$ instead based on the concept of the chain rule.

(b). If we use formulas randomly, we may obtain the following proof. Sometimes we move toward the goal; sometimes we move away from the goal. There are a limited number of formulas available in this topic, so we may have a good chance to finish the proof.

**Proof 1.**

\[ d(F_\ast g)\alpha' = \alpha'[F^* g] \quad [\text{O’Neill [74] p.154, l.5}] \]

\[ = (F(\alpha')|g] \quad [\text{by (a)}] \]

\[ = (F\alpha)'|g] \quad [\text{O’Neill [74] p.166, Exercise 13}] \]

\[ = \frac{d(g\alpha)(t)}{dt} \quad (\ast). \]

\[ F^*(dg)(\alpha') = F^*(\alpha'|g)] \quad [\text{O’Neill [74] p.154, l.5}] \]

\[ = \alpha'[gF] \quad [\text{by (a)}] \]

\[ = \frac{d(g\alpha)(t)}{dt} \quad (\ast\ast). \]

**Proof 2.** We may use (a) to prove that the second term of (\ast) equals the second term of (\ast\ast); this method will give a shorter proof than Proof 1. 

**Proof 3.** We will prove (b) based directly on the chain rule instead of (a) so that we will not get lost.

\[ F^*(dg)(\alpha') = F^*(\alpha'|g)] \quad [\text{O’Neill [74] p.154, l.5}] \]

\[ = (F\alpha)'|g] \quad [\text{O’Neill [74] p.166, Exercise 13}] \]

\[ = \frac{d(g\alpha)(t)}{dt}. \]
\[d(F,g)\alpha' = \alpha'[F^*g] \text{ [O'neill [74} \text{ p.154, 1.5]}\]
\[= \alpha'[gF] = \frac{d(F(g)(t))}{dt}.\]

VII. O’neill [74, p.166, Exercise 14].

**Proof.** (a). \(y^{-1}Fx\) and \(z^{-1}Gy\) are differentiable, so is \(z^{-1}GFx\).
(b). \(T_p(M) \xrightarrow{F} T_{F(p)}(N) \xrightarrow{G} T_{GF(p)}(P)\).
\(G,F,\alpha' = G_* (F\alpha')\) \[\text{ [O'neill [74} \text{ p.166, Exercise 13]}\]
\[= (GF)\alpha' \text{ [O'neill [74} \text{ p.166, Exercise 13]}\]. Hence
\(G_*F_* = (GF)_*\).

VIII. O’neill [74, p.161, Theorem 5.4].

**Proof.** A. \((y^{-1}Fx)_* = (y^{-1})_* (F\alpha)_* \text{ [O'neill [74} \text{ p.41, Exercise 12(c)]}\]
\[= (y)_*^{-1}F_* \alpha_\ast \text{ [O'neill [74} \text{ p.166, Exercise 14(b)]} \text{ is one-to-one.}\]
By O’neill [74, p.39, Theorem 7.10], \(y^{-1}Fx: x^{-1}\mathcal{U} \rightarrow y^{-1}\mathcal{V}\) is a diffeomorphism.
B. \(y^{-1}Fx : x^{-1}\mathcal{U} \rightarrow y^{-1}\mathcal{V}\) is differentiable [by A], i.e.,
\(F : \mathcal{U} \rightarrow \mathcal{V}\) is differentiable.
\(x^{-1}F^{-1}y = (y^{-1}Fx)^{-1} : y^{-1}\mathcal{V} \rightarrow x^{-1}\mathcal{U}\) is differentiable [by A], i.e.,
\(F^{-1} : \mathcal{V} \rightarrow \mathcal{U}\) is differentiable.

IX. O’neill [74, p.151, Exercise 13].

**Proof.** I. If \(x : D \rightarrow M\) is an arbitrary patch in \(M\), then \(x(D)\) is open in \(M\).

**Proof.** Fix \((u_0,v_0) \in D\). By O’neill [74, p.161, Theorem 5.4], there exist a \(\mathcal{U}_{(u_0,v_0)} \in \mathcal{W}\) open in \(D\) and a \(\mathcal{V}_{x(u_0,v_0)} \in \mathcal{Y}\) open in \(M\) such that \(x : \mathcal{U} \rightarrow \mathcal{V}\) is a diffeomorphism. Consequently, for every \((u_0,v_0) \in D\), there exists an open set \(\mathcal{V}\) in \(M\) such that \(x(u_0,v_0) \in \mathcal{V} \subset x(D)\).

II. Let \(y : E \rightarrow M\) be a proper patch. Then \(y : E \rightarrow y(E)\) is a homeomorphism [by definition]. Hence
\((U\text{ is open in }E) \Rightarrow (y(U)\text{ is open in }y(E))\). By I, \(y(E)\) is open in \(M\). Therefore, \(y(U)\) is open in \(M\).

Remark. O’neill [74, p.161, Theorem 5.4] is the key to solving to O’neill [74, p.151, Exercise 13; p.152, Exercise 14].
X. Let \(x, y\) be patches in \(M\). Then \(x^{-1}y\) is differentiable [O’neill [74, p.185, 1.–7]].

**Proof.** \(\frac{\partial \tilde{a}}{\partial u}, \frac{\partial \tilde{a}}{\partial v}, \frac{\partial \tilde{a}}{\partial w}, \frac{\partial \tilde{a}}{\partial z}\) are differentiable [O’neill [74, p.149, Exercise 3(b)]]. Consequently,
\(x^{-1}y : (u,v) \rightarrow (\tilde{u}(u,v), \tilde{v}(u,v))\) is differentiable in \((u,v)\).
Want to prove that \(\tilde{x}^{-1}\tilde{y} : (u,v,w,z) \rightarrow (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})\) is differentiable.
Proof. \( w_y(u,v) + z_y(u,v) = \bar{w}_x(\bar{u},\bar{v}) + \bar{z}_x(\bar{u},\bar{v}), \) where
\[
y_y = \frac{\partial y}{\partial u} \bar{x}_u + \frac{\partial y}{\partial u} \bar{x}_v, \quad y_v = \frac{\partial y}{\partial v} \bar{x}_u + \frac{\partial y}{\partial v} \bar{x}_v \quad [\text{O'neill} [74, p.149, Exercise 3(b)]]. \] Then \( \bar{w} = w \frac{\partial \bar{u}}{\partial u} + z \frac{\partial \bar{v}}{\partial v}, \bar{z} = w \frac{\partial \bar{v}}{\partial u} + z \frac{\partial \bar{u}}{\partial v} \) are differentiable.

**Example 6.86.** (Name justification for elliptic, parabolic, and hyperbolic points)

The column 4 of the table given in Kreyszig [61, p.85] should be replaced with

<table>
<thead>
<tr>
<th>Name</th>
<th>Cf. Fig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliptic point</td>
<td>25.1</td>
</tr>
<tr>
<td>Parabolic point</td>
<td>25.2</td>
</tr>
<tr>
<td>Hyperbolic point</td>
<td>25.3</td>
</tr>
</tbody>
</table>

Criticism of Kreyszig [61, p.85, Table, column 4; §28]: Examples of figure are not proofs; Kreyszig [61, §28] fails to point out the important relation \( r^2 = 2hp \) [Weatherburn [110, vol. 1, p.74, l.4–l.7]], where \( (r, \psi) \) denotes the polar coordinates of points on the intersection curve of the cutting plane.

Criticism of O’neill [74, p.204, l.6–l.14; p.398, l.8–l.9]: The features of elliptic and hyperbolic points are not fully explored.

Criticism of Weatherburn [110, vol. 1, p.75, l.8–l.7]: The feature is described, but not proved.

**Example 6.87.** (Motivation for proving a theorem about tensors)

The properties of symmetry and skew-symmetry are independent of the particular choice of coordinates [Kreyszig [61, p.111, l.12–l.11]].

I. By a linear algebra approach, we may treat a symmetric tensor as a symmetric linear transformation \( B \).

If, for one basis \( \{e_i\} \), the corresponding matrix of \( B \) is symmetric, then the corresponding matrix of \( B \) is also symmetric for any other basis \( \{u_i\} \).

Proof. \( u_i = \sum a_{im} e_m \),
\[
(Bu_i, u_j) = \sum_{m,n} a_{im} a_{jn} (Be_m, e_n).
(Bu_j, u_i) = \sum_{n,m} a_{jm} a_{im} (Be_m, e_m).
\]

Remark. This approach inspires the general direction that we should take but not details. This is because the coordinate change formula for tensors and the coordinate change formula for linear transformations look different. Actually, we may use Kreyszig [61, p.121, (36.2′); p.122, (36.5′)] to identify the difference, but we do not want to go that far. By the way, based on the knowledge given in Kreyszig [61, §36], we should rename “contravariant” as “basis-variant” and “covariant” as “dual-basis-variant”; this way is less confusing.

II. A differential geometry approach may inspires the details because the coordinate change formula for tensors and the coordinate change formula for shape operators coincide.

Let \( e_i = x_{a_i}, e_2 = x_v \) and \( B \) be the shape operator \( S \).
\[
y_u = \frac{\partial y}{\partial u} x_u + \frac{\partial y}{\partial u} x_v, \quad y_v = \frac{\partial y}{\partial v} x_u + \frac{\partial y}{\partial v} x_v \quad [\text{O'neill} [74, p.149, Exercise 3(b)].
\]

Let \( s_{12} = (Sx_u, x_v) \) and \( s_{21} = (Sx_v, x_u) \); \( s_{12} = (Sy_u, y_v) \) and \( s_{21} = (Sy_v, y_u) \).

Want to prove \( (s_{12} = s_{21}) \Rightarrow (s_{12} = s_{21}). \)
Remark. (***) can also be obtained from Kreyszig [61, p.108, (31.2b)].

III. (The tensor version) \((a_{jk} = a_{kj}) \Rightarrow (\bar{a}_{mp} = \bar{a}_{pm}).\)

Proof. \(\bar{a}_{mp} = a_{jk} \frac{\partial u^l}{\partial p^n} \frac{\partial u^k}{\partial p^m} [\text{Kreyszig [61, p.108, (31.2a)]}]\)

\[= a_{kj} \frac{\partial u^l}{\partial p^m} \frac{\partial u^j}{\partial p^n} \text{ [by hyperposis]}\]

\[= a_{jk} \frac{\partial u^l}{\partial p^m} \frac{\partial u^j}{\partial p^n} \text{ [interchage } j \text{ and } k]\]

\[= \bar{a}_{pm} [\text{Kreyszig [61, p.108, (31.2a)]}].\]

Remark. It is heuristic and insightful to connect symmetric tensors with symmetric matrices and shape operators.

**Example 6.88.** (Advantages of tensors)

In the elementary level of differential geometry, we do not need tensors. See O’Neill [73]. It even seems to look burdensome if we write an elementary theorem in tensor form. Compare Weatherburn [110] vol. 1, p.53, (2) with Kreyszig [61, p.68, (20.5); p.71, Theorem 20.3, Theorem 20.4]. Although tensors seem to be superfluous in elementary differential geometry, but they are indispensable in complicated theorems. In order to understand the advantages of tensor we should know what kind of difficulties we may encounter if we fail to use tensors.

I. Let us compare Weatherburn [110] vol. 1, p.90, (2), (3); p.91, (4)) are reduced to the two formulas given in Kreyszig [61, p.127, (38.4), (38.6)]. The four proofs of Blaga [11] p.165, (4.17.14) are reduced to one proof of Kreyszig [61, p.134, (41.5)]; the two proofs of Blaga [11] p.165, (4.17.15) are reduced to one proof of Kreyszig [61, p.134, (41.6)].

II. Avoid redundancy: The proofs of Weatherburn [110] vol. 1, p.90, (2), (3); p.91, (4)) are the same.

3. The information in tensor form is well-organized so that we may handle complicated cases easily.

4. Let us check if the computations of the above two approaches agree with each other.

\(g_{11} \Gamma^1_{22} + g_{12} \Gamma^2_{22}\)

\[= \frac{2}{\bar{\eta}^2} (2GF_2 - GG_1 - FG_2) + \frac{2}{\bar{\eta}^2} (EG_2 - 2FF_2 + FG_1) [\text{Weatherburn [110, vol. 1, p.91, (4)]}]

\[= F_2 - \frac{1}{2} G_1 = \frac{1}{2} (2 \frac{\partial g_{12}}{\partial u^1} \frac{\partial g_{12}}{\partial u^2})\]

\[= \Gamma_{222} [\text{Kreyszig [61, p.127, (38.6)]}].\]

5. Tensor and their local coordinates are two sides of one body. If Carmo were familiar with tensors, he would not have used dichotomy to separate the Gauss map from its local coordinate system [Carmo [18 p.154, 1.9–1.9; 1.5–1.5; 1.3]]. This is because the Gauss map is a tensor, and a tensor and its local coordinates are the two sides and one body. If Carmo [18] were to use tensors, Carmo [18 §3.2 & §3.3] could be incorporated into one.

6. Tensor notation makes it easier to compute, to trace origins and to link concepts, so tensor is a good tool to keep the description of things clear, concise, and complete, especially in complicated situations. By using tensor notations [Blaga [11 p.167, 1.5–1.68, 1.6]), it takes only three pages to prove Bonnet’s fundamental theorem of surface theory [Blaga [11 p.168, Theorem]]. Without using tensors, the proof would take a lot more pages and look more confusing.

II. Weatherburn [110] vol. 1, p.110, (20)) is the Beltrami’s formula for geodesic curvature [http://www.solitaryroad.com/c335a.htm]. The latter uses the Christoffel symbols, so its proof and outlook
are simple and straightforward. In contrast, the former fails to use the Christoffel symbols, so its proof and outlook look complicated and disorganized.

**Example 6.89.** (Interpretations from two different aspects of the same entity)

For integration, O’Neill [74, p.95, Fig. 2.25] and Wangness [106, p.33, Figure 1-40] illustrate the volume element in terms of spherical coordinates. For 1-dimension, \(dx\) represents an increment. The volume element is a well-defined physical quantity rather than a reliable nonsense [O’Neill [74, p.95, l.9]]. From the aspect of differentiation, in \(E^3\),

\[
d f[v] = \frac{\partial f}{\partial x_1} v_1 + \frac{\partial f}{\partial x_2} v_2 + \frac{\partial f}{\partial x_3} v_3 [\text{O’Neill [74, p.22, l.13; p.23, l.} - 13]\]

\(= v[f]\) [O’Neill [74, p.12, Lemma 3.2]].

Consequently, \(df\) can represent a 1-form whose value is a directional derivative. The statement given in O’Neill [74, p.95, l.11-l.12] is not a coincidence. Rather, it gives another interpretation of the same entity (differential) from a different aspect.

**Example 6.90.** (From the Frenet formula to the connection equations [O’Neill [74, p.87, l.7–l.14]])

I. The above two sets of formulas are based on the same idea: they express the rate of change of the frame in term of the frame itself [O’Neill [74, p.81, l.−11]].

II. The connection equations result from a multi-directional generation.

A. From the Frenet frame \(T, N, B\) to the general frame \(E_1, E_2, E_3\)

B. From a vector [O’Neill [74, p.85, l.16–l.18]] to a vector field [O’Neill [74, p.87, l.8–l.10]]


D. Frenet formulas measure the rate of change of the frame field \(T, N, B\) in the direction of \(T\), so their coefficients are real functions. In contrast, connection equations measure the rate of change of the frame field \(E_1, E_2, E_3\) in the direction of an arbitrary vector field \(V\), so their coefficients are 1-forms.

III. How to fish geometric information—the intrinsic property of geometric information

Based on the experience of studying curves, Frenet finds that geometric information often comes as a set if we adopt a frame wisely. Without using the entire set of frame, it would become difficult for us to clarify the meaning of a geometric object (e.g. Definition of geodesic curvature given in Weatherburn [110, p.109, l.9]), to organize collected information, or to get the big picture. If we use the Frenet frame as a fishing net, we can easily obtain all the useful information about a curve. A point on a curve has only one direction, while a point on a surface has many directions. To correct this problem, based on IIA & B, Élie Cartan designs another fishing net for surface [connection equations; O’Neill [74, p.248, l.9]]. Only through connection equations may the discussion on various curvatures and the torsion be clear, teamlike and complete [O’Neill [74] pp.230–231, Exercise 7; p.250, Exercise 1]].

IV. Applications

Let \(E_1 = T, E_2 = V, E_3 = U\) [O’Neill [74] p.230, Exercise 7]. By O’Neill [74] p.250, Exercise 1(b), the coefficients of connection equations directly provide the algebraic definition of geodesic curvature [O’Neill [74] p.231, l.1] and the expression for shape operator [O’Neill [74] p.248, Corollary 1.5]. This definition of geodesic curvature with its intuitive meaning given in Kreyszig [61] p.138, Fig. 42.1] is easier to understand than the one given in Weatherburn [110] p.109, l.9].

For the fundamental theorem of the local theory of curves [Carmo [18] p.19, l.−10–l.−4]], the proof of existence part given in Carmo [18] p.309, l.1–p.311, l.9] and that in Blaga [11] §1.14.3] are essentially the same: they both apply the Frenet frame [O’Neill [74] p.57, l.−9–l.−8]] to connection equations to prove the existence of curve using Pontryagin [81] p.22, Theorem 3]. The latter proof is clearer and more organized than the former one. The “columns” of the matrix \(X(s_0)\) given in Blaga [11] p.72, l.−8] should have been corrected as “rows”. The proof of uniqueness part given in Carmo [18] p.20, l.3–p.21, l.6] uses the method of calculus [Carmo [18] p.20, l.−6–l.−2]], while that given in Blaga [11] p.70, l.−4–p.71, l.−3] uses the method
of ordinary differential equations [Pontryagin [81, p.20, Theorem 2]]. Since the former proof searches for solutions in the elementary level of calculus, while the latter proof searches for solutions in the advanced level of ordinary differential equations, the former method is more effective.

It is misleading to say that connection equations or structural equations contain a lot of information about a curve or surface [O’neill [74, p.57, 1.−9–l.−8; p.96, 1.−6–l.−4; p.249, l.−2]] because the results of these equations depend on the geometric situation. The equations themselves do not contain any information. However, if one uses them by setting \( E_1 = T, E_2 = V, E_3 = U \), one can obtain \( g, k, t \) as indicated in O’neill [74] p.230, Exercise 7. Therefore, the connection equations should be treated as a useful fishing tool for geometric information.

**Example 6.91.** (Motivation vs. verification)

If a problem is given and we do not know what the answer is in advance, then the motivation to figure out the answer is required. If the answer is given [O’neill [74, p.254, l.13–l.12], then we just need a verification [O’neill [74, p.254, l.11–l.10]]. However, for a mathematician, asking from where the answer comes is always more interesting than simply verifying the answer. The formulas given in O’neill [74, p.254, l.13–l.12] are derived as follows:

**Proof.** Let \( V_1 = aF_1 + bF_2 \) and \( V_2 = cF_1 + dF_2 \).

\[ k_1V_1 = S(V_1) = (aS_{11} + bS_{21})F_1 + (aS_{12} + bS_{22})F_2. \]

By \( F_1 = \frac{ad − bc}{a} \) and \( F_2 = \frac{−cV_1 + dv_2}{ad − bc} \), we have

\[ \frac{(aS_{11} + bS_{21})(a) + (aS_{12} + bS_{22})(−c)}{ad − bc} = k_1 \] and \( (aS_{11} + bS_{21})(−b) + (aS_{12} + bS_{22})a = 0. \) Thus, \( k_1 \neq 0 \).

**Example 6.92.** (A proof should be natural and straightforward: one should not make a great fuss about little things)

The proof of Kreyszig [61, p.206, Theorem 66.1] is natural and straightforward. In contrast, the proof given in O’neill [74, p.257, 1.−8–p.259, l.13] uses two big theorems: O’neill [74, p.255, Theorem 2.7](O’neill [74, p.257, l.−3–l.−2]) and O’neill [74, p.179, Theorem 7.6](O’neill [74, p.259, l.11]). The former big theorem uses connection equations which can be avoided by Kreyszig [61, p.206, l.−8]; the proof of the latter big theorem uses reduction to absurdity which can be avoided by using the definition of compactness [Kreyszig [61, p.207, l.1–l.3]]. These two big theorems may easily distract us from the theme of Kreyszig [61, p.206, Theorem 66.1].

Because O’neill fails to explain why every point in \( M \) is in such a region \( \Theta \), I prove O’neill [74, p.257, Lemma 3.2] as follows:

**Proof.** I. Let \( \Theta \) be the range of a coordinate patch in \( M \) [O’neill [74, p.124, Definition 1.1]].

By O’neill [74, p.178, Theorem 7.5], \( \Theta \) is orientable.

By O’neill [74, p.246, Lemma 1.2], there exists an adapted frame field on \( \Theta \).

II. Let \( p \) be a fixed point in \( \Theta \), \( q \) be an arbitrary point in \( \Theta \), and \( \alpha \) be a curve in \( \Theta \) such that \( \alpha(0) = p \) and \( \alpha(1) = q \).

\[ \frac{dK(\alpha(t))}{dt} = \left[ K[O’neill [74, p.149, Definition 3.10]] \frac{dK[\alpha'(t)]}{dt} \right] [O’neill [74, p.23, Definition 5.2]] = 0 \] [O’neill [74, p.258, l.1–l.2]]. Thus, \( K \) is constant on \( \alpha[0, 1] \). In particular, \( K(p) = K(q) \).

III. Let \( q \) be an arbitrary point in \( M \) and
\[ \beta \text{ be a curve in } M \text{ such that } \beta(0) = p \text{ and } \beta(1) = q. \]

\[ \beta[0,1] \text{ is compact. Consequently, there exist a finite number of } \mathcal{E}_i \text{ such that } \beta[0,1] \subset \bigcup_i \mathcal{E}_i. \]

By II, \( K \) is constant on each of \( \{ \mathcal{E}_i \} \). Therefore, \( K \) is constant on \( \beta[0,1] \).

Remark. We assume that the surface is of class \( r \geq 3 \) because we use the partial derivatives of \( \lambda \) in Kreyszig [61 p.206, (66.3)].

**Example 6.93.** (Take one thing at a time)
\[
| \sum (\alpha(t_i - t_{i-1})| + \sum (t_i - t_{i-1})|\alpha'(t_i)|) |
\leq | \sum (t_i - t_{i-1})| \sup_i |\alpha'(s_i)| - | \sum (t_i - t_{i-1})|\alpha'(t_i)| |
\leq | \sum (t_i - t_{i-1})| \sup_i |\alpha'(s_i) - \alpha'(t_i)| |
\leq \frac{1}{2} \text{ [Carmo 18 p.475, 1–4–1–2]}. 
\]

**Proof.** The first inequality follows from Rudin [86 p.99, Theorem 5.20]; the second inequality follows from the following inequalities:
\[
0 \leq \sup_i |\alpha'(s_i) - |\alpha'(t_i)| |
\leq \sup_i |(\alpha'(s_i) - \alpha'(t_i))| \leq \sup_i |\alpha'(s_i) - \alpha'(t_i)|. 
\]

Remark. Rudin [86 p.125, Theorem 6.35] provides a step-by-step proof of Carmo [18 p.475, Solution to §1-3, Exercise 8]. The former proof is clearer because Rudin considers the subinterval \([x_{i-1}, x_i]\) first [Rudin [86 p.125, 1–6–1–1]]. In contrast, the latter proof is confusing because Carmo attempts to omit this first step and jump directly to sum over the partition of the entire interval \([a, b]\). Furthermore, to prove \(|A| \leq |B|\), we should let \( C = |B| \) and divide the proof into two parts: Part I. \( A \leq C \); Part II. \( -A \leq C \). If we know \( B > 0 \), we should not write \( B \) as \( |B| \). For the expression given in Carmo [18 p.475, 1–3], we should write it as \( \sum (t_i - t_{i-1}) \sup_i |\alpha'(s_i) - |\alpha'(t_i)| | \) rather than \( \sum (t_i - t_{i-1}) \sup_i |\alpha'(s_i) - \sum (t_{i-1} - t_i)\alpha'(t_i)|. \)

**Example 6.94.** (Definition’s accessibility and effectiveness: Differentiable functions on a regular surface)

In order to show Carmo [18 p.72, 1.10–1.14, Definition 1] does not depend on the choice of parametrization [Carmo [18 p.72, 1.15–1.18]], we have to prove Carmo [18 p.70, Proposition 1] first. The inverse function theorem given in Carmo [18 p.71, 1–3] refers to O’Neill [74 p.39, Theorem 7.10] \([C^\infty\text{-version}]\) which is different from Rudin [86 p.193, Theorem 9.17] \([C^1\text{-version}]\). However, O’Neill [74 p.39, Theorem 7.10] follows from Rudin [86 p.193, Theorem 9.17(b)] and Jacobson [56 vol. 1, p.59, (9)] because the elements of the matrix \( dF^{-1} \) can be calculated using the elements of the matrix \( dF \). I cannot locate the place where the continuity of \( x^{-1} \) is used in the proof of Carmo [18 p.70, Proposition 1] [see Carmo [18 p.72, 1–2–1–1]], but I can locate the place where the continuity of \( r^{-1} \) is used in the proof of Blaga [11 p.119, Theorem 4.3.1] which is the same as Carmo [18 p.70, Proposition 1]: see Blaga [11 p.120, 1–10–1–9] in the proof of Blaga [11 p.120, Lemma]. By comparison, the proof of Blaga [11 p.119, Theorem 4.3.1] is simpler than that of Carmo [18 p.70, Proposition 1] because Blaga [11 p.120, 1–14–1–9] uses the \( n = 2 \) version of the inverse function theorem, while Carmo [18 p.71, 1–3–1–10] uses the \( n = 3 \) version. Blaga [11 p.120, Lemma] is useful; the proofs of the differentiability of \( x^{-1} \) [O’Neill [74 p.144, 1.12]], O’Neill [74 p.145, Theorem 3.2], and Carmo [18 p.70, Proposition 1] are artificial and far-fetched, but if we use Blaga [11 p.120, Lemma] to prove them, then the proofs become natural and perfect.

For application, the conditions given in a definition should be simple and there should be an easy and effective way to check the definition. Let us compare Carmo [18 p.72, Definition 1] with O’Neill [74 p.143,
The former definition uses the word “some” [Carmo [18, p.72, l.12]], while the latter definition uses the word “all” [O’neill [74, p.143, l.8]]. Consequently, the former definition is better because once one finds a single parametrization satisfying the conditions, it is unnecessary to check any other parametrization whose range containing $p$. The definition of atlas [Spivak [95, vol. 1, p.37, l.4]] is just smoke and mirrors to avoid using the inverse function theorem, Carmo [18, p.70, Proposition 1], or other methods to solve problems in special cases. A definition does not require a proof, but this does not mean that a definition can be used as a black-box-warehouse for piling up unsolvable problems even though they may be solvable in special cases. In the domain that we have no way to exercise our judgment, or for the verifying procedure that requires infinite steps, how can we tell whether a statement is true or not? This is not the end of the story; after defining a $C^\infty$-manifold [Spivak [95, vol. 1, p.38, l.11]], to reach the definition of a regular surface still has a long way to go [Kreyszig [61, §16–§17]]. Thus, this type of definition of a regular surface is wrapped in multi-levels of black boxes. I do not think it could conjure up any useful images. Furthermore, a man who does not want to ask for trouble would prefer $\{I\}$ to $U$ [Spivak [95, vol. 1, p.38, l.13–l.14]] when testing the differentiability of a function on a differentiable manifold.

O’neill [74, p.124, l.3–l.11; p.182, l.4–l.18; p.184, l.5–l.15] discuss the goal, motivation, and the method of generalization in constructing surfaces or differentiable manifolds. For the definition of abstract surface, we ignore the logical impossibility [O’neill [74, p.182, l.17–l.18]] of effective testing the definition and try to treat the conditions as an axiom for the structure of surface [O’neill [74, p.182, l.18]]. For raising effectiveness, Carmo [18, p.70, Proposition 1] can be proved as a theorem instead of being treated as an axiom [O’neill [74, p.182, Definition 8.1(2)]] using O’neill [74, p.125, Definition 1.2]. In the beginning stage of defining a $C^\infty$-manifold, Spivak introduces conditions one at a time based on needs [Spivak [95, vol. 1, p.35, l.10; l.1]] but, as he introduces atlas and the maximal atlas, the theory’s quality deteriorates because some unnecessary considerations [Spivak [95, vol. 1, p.37, l.2–l.4; p.37, l.8–p.38, l.4]] are thrown into the definition. His excuse is saving words [Spivak [95, vol. 1, p.37, l.1]]. In fact, his formulation is simple in words, but complicated in thoughts, while O’neill [74, p.184, Definition 8.4] is simple in thoughts because it is a lean and mean generalization from O’neill [74, p.125, Definition 1.2].

Remark. The definition given in Carmo [18, p.73, l.6–l.1] shows its close and strong connection with the primitive model given in O’neill [74, p.124, l.1–l.7], while the two-step approach given in Blaga [11, p.132, Definition 4.7.1; p.133, Definition 4.7.2] blurs this connection.

**Example 6.95. (Theory vs. application)**

In the study of mathematics, theory tends to deal with the difficult parts, especially the hidden one; while user-friendly applications tend to avoid dealing with the difficult parts. The first statements in O’neill [74, p.151, Exercise 13] follow from O’neill [74, p.161, Theorem 5.4]. The proof of Carmo [18, p.70, Proposition 1] fails to mind this detail, so it becomes difficult to locate the place where the continuity of $x^{-1}$ is used in the proof [Carmo [18, p.72, l.2–l.1]]. Similarly, Blaga [11, p.120, Lemma] fails to show that $W$ is open in $S$, so the statement given in Blaga [11, p.120, l.10–l.9] becomes difficult to understand.

From the viewpoint of application, theorems in a theory are like packages or software products designed for completing a mission in an easy way. A user-friendly software product absorbs all difficult parts so that users need not touch them. To get the result, all one has to do is put data into it. If one tries to prove a statement by definition instead of applicable theorems, there would be lots of odds and ends to deal with and one would inevitably bump into the difficulties in reconstructing the theory. In order to prove that a surface of revolution is a surface, O’neill [74, p.129, Example 1.6] uses O’neill [74, p.127, Theorem 1.4]. All one has to do is show that $dg$ is never zero on $M$ [O’neill [74, p.130, l.6]]. In contrast, Carmo [18, p.76, Example 4] proves this fact by definition, so the proof contains lots of odds and ends. The proof of the statement given in Carmo [18, p.77, l.1–l.3] requires the use of the difficult inverse function theorem. The failure to mention
this makes the statement difficult to understand.

Sometimes a theorem has many versions: Spivak \cite[p.41, (3)]{95} & Blaga \cite[p.120, Lemma]{11}. They have different functions. In application, the version given in Spivak \cite[p.41, (3)]{95} is used most often. In contrast, the version given in Blaga \cite[p.120, Lemma]{11} points out directly the key to the proof of the theorem.

**Example 6.96.** (In order to keep up with the modern research, we should adopt a new viewpoint toward the inverse function theorem)

The inverse function theorem usually refers to the version given in O’neill \cite[p.161, Theorem 5.4]{74}. In order to keep up with the modern research, we should adopt a new viewpoint toward the inverse function theorem and interpret it as the following natural and complete version that can be illustrated by a geometric figure (i.e., a linear isomorphism between tangent spaces is equivalent to a diffeomorphism between coordinate neighborhoods). For the proof of O’neill \cite[p.161, Theorem 5.4]{74}, most authors of differential geometry textbooks pass the buck to the authors of advanced calculus textbooks. However, the students who have studied advanced calculus may still not know where to start the proof because mappings of surfaces are more complicated than mappings of open sets in planes. The difficulty lies in the attempt to fit many requirements. Before I start the proof, I point out two facts: Spivak \cite[p.41, (1)]{95} and the chain rule \cite[p.73, l.6]{18}.

**(The inverse function theorem)** Let \(F: M \to N\) be a mapping of surfaces and \(p \in M\). Then \((F_p : T_p(M) \to T_{F(p)}(N))\) is a linear isomorphism at \(p \in M\) \[O’neill \cite[p.161, Theorem 5.4]{74}\].

**Proof.** \(\Leftarrow: F^{-1} \circ F = I \Rightarrow d(F^{-1}) \circ dF = I\) \[Carmo \cite[p.91, Exercise 24]{18}\].

\(\Rightarrow: \) By Spivak \cite[p.56, Theorem 9(1)]{95}, \(y \circ F \circ x^{-1} : U \to V\) is a diffeomorphism. The result follows from Spivak \cite[p.41, (1) & (3)]{95}. \(\square\)

**Remark 1.** \(\det \left[ \frac{\partial x^i}{\partial u^j}(p) \right] \neq 0; \) by the \(\mathbb{R}^n\)-version of the inverse function theorem, there are a neighborhood \(W_1\) of \(u(p)\) and a neighborhood \(W_2\) of \(x(p)\) such that \(x \circ u^{-1}\) is a diffeomorphism from \(W_1\) onto \(W_2\) \[Spivak \cite[p.57, l.6–l.8]{95}\].

**Remark 2.** The word “square” given in Spivak \cite[p.57, l.1–l.1]{95} should have been replaced with “rectangle”; \(D_m \psi^m\) given in Spivak \cite[p.57, l.1–l.1]{95} should have been replaced with \(D_n \psi^m\).

**Remark 3.** By Rudin \cite[p.196, Theorem 9.20]{86}, we can write \(\psi^r(a) = \psi^r(a^1, \cdots, a^k), r = k + 1, \cdots, m\) \[Spivak \cite[p.58, l.1–l.2]{95}\]. Note that \(\psi^r \in C^\infty\).

**Remark 4.** For the proof of Spivak \cite[p.59, Theorem 10(2)]{95}, we may incorporate the proof for the the special case \(x = I\) and that for the general case into one.

**Example 6.97.** (How a mathematical passage should be formulated)

Writing a mathematical passage should not be like opening Pahdora’s box or listing a bunch of statements whose truth needs to be validated. A passage should
(1). have a central concept, a central theme, and a key statement,
(2). take an advantageous viewpoint that may broaden our vision or make us see clearly the role played by each individual in the overall situation,
(3). be structured in levels; more precisely, be proceeded from the central theme outward level by level, and
(4). be organized in a systematic way so that the entire passage circles around the central theme. Otherwise, a disorganized passage only shows that its author fails to master that topic.

The passage to be discussed: Carmo \cite[p.214, l.12–p.215, l.4]{18}.
The central concept of this passage: self-adjoint linear operators.
The center theme of this passage: The trinity of the self-adjoint linear operators, quadratic forms, and bilinear symmetric forms.

The key statement of this passage: Every quadratic form \( Q(u) = ax^2 + 2bxy + cy^2 \) can be expressed as \((x,y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \langle Au, u \rangle\), where \(A\) is a symmetric matrix.

How we use the key statement to establish the trinity:
\[ Q(u) = \langle Au, u \rangle; \]
\[ B(u, v) = \langle Au, v \rangle. \]

The environment is like a dark room and pointing out the central theme is like turning its lights on. Carmo fails to point out the key statement, but improperly emphasizes the relation given in Carmo [18, p.215, l.2]. This formula makes proving the bilinearity of \(B(u, v)\) difficult, thereby, the exact meaning of the statement in Carmo [18, p.215, l.−4] becomes unclear.

**Example 6.98.** (A definition should avoid using abstract concepts and strange symbols [e.g., \(\nabla_v\)] and use concrete concepts and traditional methods)

The concept of differential of Gauss map in Carmo [18, p.136, Definition 1] and that of shape operator in O’Neill [74, p.190, Definition 1.1] are essentially the same. However, Carmo adopts a concrete and traditional approach, while O’Neill adopts an abstract [e.g., covariant derivatives] and unconventional [e.g., the minus sign in O’Neill [74, p.190, l.−2–l.−1]] approach. Because Carmo uses the concept of differential of a map, the mathematical context of differential of Gauss map becomes crystal clear and we may directly quote a known theory [Carmo [18, p.84, Proposition 2]] instead of rebuilding it (e.g., if we use the former definition, it is unnecessary to prove O’Neill [74, p.191, Lemma 1.2]. All we need to do is identify \(T_{N(p)}(S^2)\) with \(T_p(S)\) [Carmo [18, p.136, l.−11–l.−9]]). The proof given in Carmo [18, p.138, l.2–l.8] is easy to understand, while the proof given in O’Neill [74, p.191, l.−1–p.192, l.4] is not, because \(p + tv\) in O’Neill [74, p.78, l.2] may not be a curve on the surface.

An abstract concept is often obtained by cutting a piece from the whole, breaking its outside links, weakening its effectiveness, emptying its contents, and considering it as a closed system of its own. The concept of covariant derivative is used in O’Neill [74, p.190, Definition 1.1]. Thus, we may say that the concept originates from the study of surface. However, according to O’Neill [74, pp.77–78, Definition 5.1; FIG. 2.20], covariant derivatives can be defined without referring to any surface. What we have learned in O’Neill [74, §II.5] is just a black-box mechanism without any application. The gain of this mechanism is at the cost of insights, motivations, and the big picture. A definition obtained by assembling a series of such black-box mechanisms will not help visualize its geometric image.

**Example 6.99.** (Conjugate directions at an elliptic or hyperbolic point of a surface)
I. The origin.
A. In \(\mathbb{R}^3\), in terms of diametrical planes, the conjugate diameters are given by Bell [6, p.115, Fig. 36; (A’), (B’T)]. In \(\mathbb{R}^2\), the above statement is reduced to the following:
B. In terms of diametrical lines, the conjugate diameters are given by Kreyszig [61, p.176, (56.2)].

II. The characterizations of conjugate directions in various aspects.
A. In terms of developable surfaces having contact with a surface \(S\) along a curve on \(S\), the conjugate directions are given by Weatherburn [110, vol. 1, p.80, (17)].
B. Weatherburn [110, vol. 1, p.80, (17)] is the same as Kreyszig [61, p.176, (56.3)] (in terms of the second fundamental form).
C. Kreyszig [61, p.176, (56.3’)] is the same as Carmo [18, p.150, Definition 10] (in terms of the self-
Example 6.100. (A definition of a concept should directly explain what it is in simple words first)

A definition of a concept should directly explain what it is in simple words first. If its origins and evolution history are introduced first, the readers may ask if the origins and evolution history are the indispensable parts of the definition or if it is not possible to obtain the concept by other methods. In Weatherburn [110, vol. 1, §32], the origins and evolution history of the Dupin indicatrix are introduced before its definition. In contrast, the definitions given in both Carmo [18, p.148, l.−2–l.−1] and Kreyszig [61, p.97, l.5–l.13] directly explain what the Dupin indicatrix is in simple words first. Carmo’s version is based on the second fundamental form, while Kreyszig’s version is based on Euler’s theorem. Both Carmo [18, §3-3, Example 5] and Kreyszig [61, p.98, Theorem 28.4] provide the material in Weatherburn [110, vol. 1, p.74, l.10–p.75, l.2] after the definition of the Dupin indicatrix is given.

Remark. The formula of normal curvature given in Carmo [18, p.142, l.9–l.11] looks different from that given in Kreyszig [61, p.82, (24.2)]. However, $|\alpha’(0)| = |x_uu’(0) + x_vv’(0)| = 1$, so the denominator of the fraction given in Kreyszig [61, p.82, (24.2)] is 1.

Example 6.101. (Mathematical training is to teach students the standard methods of removing obstacles)

Mathematical training is to teach students the methods of removing obstacles, especially the standard ones that will be used again and again. The more methods one has learned, the higher one’s skill and the more chances one may do some creative works in mathematics. A textbook in mathematics should be carefully written. The shortcomings of a popular one could have affected the later textbooks for centuries. A case in point is the topic on principal directions at a point on a surface. A proper method to deal with this topic is to use the standard ones in linear algebra (Prove the existence given in Carmo [18, p.144, l.10–l.9] by using Carmo [18, p.216, Theorem] which is a special case of Jacobson [56, vol. 2, p.180, Theorem 4]; the equations of Weingarten [the general case with a coordinate system: Carmo [18, p.155, (3); l.16–l.13]; special cases without a coordinate system [in terms of tangent vectors]: O’Neill [74, Exercise 9 (a) & (b)]; the fact that the principal directions are orthogonal [Carmo [18, p.161; l.6]] follows from the definition of self-adjoint linear map [Carmo [18, p.214, l.7–l.18]]). Today some textbooks have been on the right track [e.g., Blaga [11, p.156, Definition 4.16.1]], others have not [e.g., (3. 37) in http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/node30.html].

The shortcomings of Weatherburn [110, vol. 1, §29 & §30]: The material in these two sections is heuristic and intuitive, but it is difficult to make the definition of consecutive points [Weatherburn [110, vol. 1, p.66, l.7]] rigorous. However, by calculus, Weatherburn [110, vol. 1, p.69, l.3–p.70, l.7] proves that the principal directions at a point are the directions of greatest and least normal curvature. The proof of Kreyszig [61, p.91, (27.4)] provides rigorous details of the previous proof. The essence of topic lies in its geometric meanings rather than computation details. The differential of Gauss map in Carmo’s approach is the right tool for expressing clear geometric images, while the complicated matrix elements given in
Carmo [18, p.155, l.−16–l.−13] that Weatherburn [110, vol. 1, p.66, (2)] uses are simply not. Differential geometry contains a lot of topics, so the mastery of differential geometry requires a high level of organizing skills. Carmo [18, pp.167–168, Remark] points out that, with the help of Gasss map, the Gaussian curvature [Carmo [18, p.167, Proposition 2]] is the 2-dim geometric [Carmo [18, p.165, l.−7–l.−5]] analogue of the signed curvature of a plane curve [Carmo [18, p.22, Exercise 3.b]]. Although the analogy is interesting, it fails to give an incentive for introducing O’neill [74, p.191, Lemma 2]. In order to show that the shape operator is a natural 2-dim generalization of the curvature for plane curve, I would like to point out that for a plane curve, 

\[ N' \cdot T = \kappa \] 

[O’neill [74, p.58, Theorem 3.2]] is the special case of 

\[ k(u) = S(u) \cdot u \] 

[O’neill [74, p.196, Definition 2.2]] when \( u = e_1 \). In the case of surface, we let \( u \) vary. The proof of the existence of two real principal curves in Kreyszig [61, p.92, l.−8] and the proof of (3.44) in [http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/node30.html] are quite tricky, while the proof of the existence of eigenvectors of a self-adjoint linear map uses a standard method in linear algebra. The inner product notation for tensors used in the proof of the orthogonality [Krey szig [61, p.92, l.−6]] of the lines of curvature is more complicated than the corresponding notation for vectors in linear algebra.

The shortcomings of O’neill [74, p.200, Theorem 2.5 (2)]: the proof fails to take advantage of the standard method in linear algebra, so one may easily forget the argument. A more serious problem is about the existence of \( e_1 \) [O’neill [74, p.200, l.17]]. The existence is derived from reduction to absurdity [Rudin [86, p.35, l.17–l.−11; p.77, Theorem 4.15]]. There is no effective algorithm of obtaining \( e_1 \). Consequently, the method of obtaining \( e_1 \) in O’neill [74, p.200, l.17] by using compactness is less effective than the method of obtaining principal directions in the proof of Kreyszig [61, p.91, (27.4)] by using calculus [Kreyszig [61, p.90, l.−7]].

**Example 6.102.** (The case classification for Bertrand curves)

Suppose a regular curve \( C \) with \( (\kappa, \tau) \) is given. We want to find its conjugates \( C_1 \) with \( (a, \alpha) \). The curve and its conjugates are related by the formula given in Weatherburn [110, vol. 1, p.35, l.9], where \( \kappa, \tau \) are functions; \( a, \alpha \) are constants. Let us call \( \kappa, \tau, a, \alpha \) parameters. If the condition of one parameter is specified, the condition could affect other parameter values. This property makes it difficult to determine when we have completed the discussion of a case concerning a certain parameter or what remains to be done in the process of case classification.

Both the results of Weatherburn [110, vol. 1, p.36, Ex. 1] and those of Blaga [11, p.62, Corollary 1.10.2] are confusing because the conditions for \( (\kappa, \tau) \) are not accurately specified. The discussion given in Struik [98, p.43, Exercise 13] is not complete, so it fails to assemble the big picture. Consequently, when making a case classification for Bertrand curves, we should manage to not allow any case be left out.

Case I: \( \tau = 0 \).

A. By O’neill [74, p.61, Corollary 3.5], \( C \) is a plane curve.

By Blaga [11, p.62, l.3–l.5], \( C \) has an infinity of conjugates.

B. \( \alpha = 0 \) [Carmo [18, p.21, l.9]].

We have completed the discussion for the case \( \tau = 0 \). The following assumes \( \tau \neq 0 \).

Case II: \( \kappa = 0 \).

\( C \) is a straight line [O’neill [74, p.42, Lemma 3.6]].

\( C_1 \) can be any straight line parallel to \( C \).

We have completed the discussion for the case \( \kappa = 0 \). The following assumes \( \kappa \neq 0 \).

Case III: both \( \kappa \) and \( \tau \) are nonzero constant.

By O’neill [74, p.119, Corollary 5.5], \( C \) is a circular helix.

By Struik [98, p.43, Exercise 13(c)], \( C \) has an infinity of conjugates, all circular helix.

By O’neill [74, p.120, l.11], these helices are on the coaxial cylinders.
We have completed the discussion for the case: both $\kappa$ and $\tau$ are constant. The following assumes either $\kappa$ or $\tau$ is not constant.

Case IV: $\kappa$ is constant.

A. By Blaga [11, p.60, Theorem 1.10.4], $a\kappa + b\tau = 1$, where $b = a\cot\alpha$ [Blaga [11] p.61, (1.10.24)].

$$a\kappa' + b\tau' = 0.$$

$\exists s: \tau'(s) \neq 0$ [since $\tau$ is not constant]. Then $b = 0$. Hence $\alpha = \pi/2$ [since $b = a\cot\alpha$].

B. $(1 - a\kappa)\sin\alpha + a\tau\cos\alpha = 0$ [Weatherburn [110, vol. 1, p.35, l.9]].

$1 - a\kappa = 0$. Hence $a = 1/\kappa$.

C. $C$ is the locus of the centers of curvature of $C_1$.

Proof. $d\kappa = \rho\tau b$ [Weatherburn [110] vol. 1, p.17, l.-11–l.-9].

$t_1 = \frac{ds}{ds_1}\rho\tau b$.

We may let $t_1 = b$ (i.e., $\frac{ds}{ds_1} = \frac{1}{\rho\tau}$).

$$\kappa n_1 = \frac{dt_1}{ds_1} = \frac{b}{\rho \tau} = -\kappa n.$$ Thus,

$$\kappa_1 = \kappa$$ and $n_1 = -n$.

We have completed the discussion for the case: $\kappa$ is constant. The following assumes $\kappa$ is not constant.

Case V: $\tau$ is constant.

By Blaga [11] p.60, Theorem 1.10.4, $a\kappa + b\tau = 1$.

$$a\kappa' + b\tau' = 0.$$

$\exists s: \kappa'(s) \neq 0$ [since $\kappa$ is not constant]. Then $a = 0$. Hence $C_1 = C$.

We have completed the discussion for the case: $\tau$ is constant. The following assumes $\tau$ is not constant.

Case VI: both $\kappa$ and $\tau$ are not constant; $C$ has a unique conjugate.

Then the equation given in Weatherburn [110] vol. 1, p.35, l.9] has a unique solution $(a, \alpha)$ for $C_1$.

Case VII: both $\kappa$ and $\tau$ are constant. Thus, we obtain a contradiction.

Remark 1. Case II is a degenerate case of both Case I and Case III.

Remark 2. Blaga [11] p.60, (1.10.19)] is the same as the formula given in Weatherburn [110] vol. 1, p.35, l.9]

Remark 3. Both Weatherburn [110] vol. 1, p.36, l.2–l.5] and Struik [98, p.43, Exercise 15] describe Mannheim’s theorem, but only the cross-ratio notation of the latter version is consistent with the notation given in https://en.wikipedia.org/wiki/Cross-ratio.

Remark 4. Any circular helix can be used as a counterexample of both statements given in Weatherburn [110] vol. 1, p.36, l.9–l.11]. Thus, we should correct the hypothesis of the first statement as “$\kappa$ is a nonzero constant and $\tau$ is not constant” and correct the hypothesis of the second statement as “$\tau$ is a nonzero constant and $\kappa$ is not constant”. Only through the above closely linked case classification may we be able to correct these mistakes.

Example 6.103. (Intuition vs. rigor: the Möbius strip is not orientable)

Example 6.104. (Bonnet’s fundamental theorem of surface theory)
I. The theorem lays the foundation of differential geometry. Its proof uses the theory of partial differential equations. It is interesting to see that PDE theory has such a useful geometric application.
II. The discussion here is based on Blaga [11, §4.17.4]. In order to make the section readable, we correct some errors first:
\[
\begin{align*}
    F\Gamma_{11} + G\Gamma_{22} &= F_u - \frac{1}{2}E_u \quad \text{[Blaga [11, p.163, l.7—l.9]}],
    F\Gamma_{11} + G\Gamma_{22} &= F_u - \frac{1}{2}E_u, \\
    \Gamma_{22} &= \frac{EG'_{12} - 2FG'_{11}}{2(EG - F^2)} \quad \text{[Blaga [11, p.164, l.3]}],
    \Gamma_{22} &= \frac{EG'_{12} - 2FG'_{11}}{2(EG - F^2)}.
\end{align*}
\]
\[
\begin{align*}
    \Gamma_{11} &= \frac{1}{2}E_u = \frac{\partial}{\partial u} \ln E, \quad \Gamma_{11}^2 = \frac{1}{2} \frac{\partial}{\partial u} \ln G, \\
    \Gamma_{12} &= \frac{\partial}{\partial v} \ln E, \quad \Gamma_{12} = \frac{\partial}{\partial v} \ln G, \quad \text{[Blaga [11, p.164, l.9—l.11]}],
    \Gamma_{12} &= \frac{\partial}{\partial v} \ln E, \quad \Gamma_{12} = \frac{\partial}{\partial v} \ln G, \\
    \Gamma_{22} &= \frac{\partial}{\partial u} \ln G, \quad \Gamma_{22} &= \frac{\partial}{\partial v} \ln G, \\
    \Gamma_{11}' &= a_{11}r_{1u}' + a_{12}r_{1v}', \\
    \Gamma_{12}' &= a_{11}r_{1u}' + a_{12}r_{1v}', \\
    \Gamma_{22}' &= a_{11}r_{1u}' + a_{12}r_{1v}', \\
    \Gamma_{11}' &= a_{11}r_{1u}' + a_{12}r_{1v}', \\
    \Gamma_{12}' &= a_{11}r_{1u}' + a_{12}r_{1v}', \\
    \Gamma_{22}' &= a_{11}r_{1u}' + a_{12}r_{1v}'.
\end{align*}
\]
If we equate to zero the coefficients of \(r_1'^u\) follows the first of the Gauss’ equations, while from the coefficient of \(r_1'^v\) follows the second of the Gauss’ equations [Blaga [11, p.167, l.1—l.9]}. If we equate to zero the coefficients of \(r_1'^u\) follows the first of the Gauss’ equations, while from the coefficients of \(r_1'^v\) follows the second of the Gauss’ equations.
III. Blaga [11] has not introduced the concept of tensor before Blaga [11, §4.17.4]. However, Blaga [11, p.167, l.5—p.171, l.1] suddenly assumes that readers have some background on tensors. Tensor notation makes it easier to compute, to trace origins and to link concepts, so tensor is a good tool to keep the description of things clear, concise, and complete, especially in complicated situations. All the required background can be found in Kreyszig [61], so I will use Kreyszig [61] as a reference book in III.
1. \(\frac{\partial F}{\partial u} = -\sum_{i=1}^{2} l_i g_{ij} l_i r_j\), [Blaga [11, p.169, l.3, (4.17.22)]}, is Kreyszig [61, p.126], (37.1).
2. (i). By the proof of Blaga [11, p.165, Theorem 4.17.1], we obtain the compatibility conditions: Blaga [11, p.169, (4.17.23)]).
(ii). By Struik [98, p.135, Exercise 21], we obtain the compatibility conditions: Blaga [11, p.169, (4.17.23)]).
Remark. \(r_1', n\) are unknown; we are all allowed to use to prove these compatibility conditions as identities is the relations given in Blaga [11, p.169, (4.17.22)].
3. \(r_1^{(0)} \parallel r_2^{(0)} \Rightarrow n^{(0)} = (0, 0, 1)\), [Blaga [11, p.169, l.8—l.9]].
4. The functions (4.17.28) give a solution of the system (4.17.29) [Blaga [11, p.171, l.11]], which means that the system is completely integrable and hence the compatibility conditions are automatically satisfied [Blaga [11, p.171, l.1—l.2]].
5. By Kreyszig [61, p.128, (39.2)], \(\frac{\partial g_i}{\partial u} = \sum_{i=1}^{2} l_i k_{ij} g_{ij} + \sum_{i=1}^{2} l_i k_{ij} g_{ij}\), [Blaga [11, p.171, l.9]].
6. \(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \neq 0\) [Blaga [11, p.170, l.16]], \(r_1', r_2', n > 0\), [Blaga [11, p.171, l.17]].
7. By Kreyszig [61, p.80, (23.10)], \(b_{il} = -n_i r'_l\), [Blaga [11, p.171, l.2—l.2]].
1. The kernel and PDE component of Bonnet’s theorem is the following theorem:

**PDE Theorem.** Let Levi-Civita [67, p.14, (4′)] be a system of PDEs with initial conditions. If Levi-Civita [67, p.15, (5)] can be proved as local identities using the relations given in Levi-Civita [67, p.14, (4′)] alone, then there exists locally a unique set of solutions for the system.

**Remark 1.** The difficult part lies in how to interpret Levi-Civita [67, p.15, (5)]. If \( r_i' \) is known, then obviously Blaga [11, p.169, (4.17.23)(i)] \( \Rightarrow \) Blaga [11, p.169, (4.17.23)(ii)] [Blaga [11, p.127, l.−4]]. However, if \( r_i' \)'s are unknown and only subject to Blaga [11, p.169, (4.17.22)], then Blaga [11, p.169, (4.17.23)(ii)] cannot follow from Blaga [11, p.169, (4.17.23)(i)]. Blaga fails to tell the difference between these two interpretations of Levi-Civita [67, p.15, (5)], so he fails to do as in III.2(ii). I suggest that we call Levi-Civita [67, p.15, (5)] compatibility conditions as in Struik [98, p.110, l.10] when \( u \)'s are known and that we call Levi-Civita [67, p.15, (5)] integrability conditions as in Levi-Civita [67, p.18, l.10–l.11] when \( u \)'s are unknown and only subject to a system of PDEs. Hopefully, this way may clarify the confusion [e.g., Levi-Civita [67, p.18, l.12–l.14] is misleading. It should have said that \( x \)'s and \( u \)'s are subject to Levi-Civita [67, p.14, (4′)] somewhat.

**Remark 2.** My criticism on the analysis given in Levi-Civita [67, §II.2; §II.3]:

“But it may happen – and this is the most interesting case – that the equations (5) are not only satisfied for those particular \( u \)'s which form a solution of the system but are true identically, i.e., for any set of values whatever of the \( u \)'s and of the \( x \)'s.” [Levi-Civita [67, p.15, l.−7–l.−4]]

Based on Levi-Civita [67, p.17, l.6–l.12], the above passage does not seem to be what Levi-Civita had in mind. However, based on its literal sense, I have to criticize it.

**Criticism 1.**

Hypothesis W: Levi-Civita [67, p.15, (5)] can be proved to be local identities by using Levi-Civita [67, p.14, (4′)] alone.

Hypothesis S: Levi-Civita [67, p.15, (5)] are identities for arbitrary \( u \)'s and \( x \)'s.

PDE Theorem’. Given Hypothesis S. Then there exists locally a unique set of solutions for the system: Levi-Civita [67, p.14, (4′)].

By comparing the two hypotheses, PDE Theorem is stronger than PDE Theorem’

**Criticism 2.** A logician can say that Case I or Case II is true without any conditions. However, before a mathematician says this, he or she should search for an effective test and prove that these two cases are its different results. Namely, he or she should explain how to get these cases. Before the search, any discussion about the two cases is meaningless because otherwise the answer to other questions cannot be determined.

“It may happen” is simply not a phrase that a mathematician should use, ditto for “If exceptionally these conditions are mutually consistent” [Levi-Civita [67, p.16, l.13–l.14]] and “If all these reduce to identities” [Levi-Civita [67, p.17, l.13]].

2. The important steps for the proof of PDE Theorem given in 1.

(i). Given a line \( T[x_i = \phi_i(t)] \) from \( P_0 \) to \( P_1 \). We may reduce Levi-Civita [67, p.14, (4′)] to Levi-Civita [67, p.23, (14) or (14′)]. By the uniqueness of solutions of a system of ODEs, we may obtain a unique solution of Levi-Civita [67, p.23, (14) or (14′)].

(ii). Levi-Civita [67, p.15, (5)]

\[ \Rightarrow \] Levi-Civita [67, p.21, (12)] [by Levi-Civita [67, p.20, (11)]]

\[ \Rightarrow \] Levi-Civita [67, p.24, (16)]

\[ \Rightarrow \] Levi-Civita [67, p.24, (17)]

\[ \Rightarrow \] Solutions of Levi-Civita [67, p.23, (14) or (14′)] are consistent in a simply connected domain.

3. Levi-Civita [67, p.19, l.8–l.11; l.16–l.20] should have been replaced with the following statement:

Without loss of generality, we may assume \( d\delta x_i = \delta dx_i (i = 1, 2, \cdots, n) \) (9).

\[ d\delta x = d(\epsilon \chi(t)) = \epsilon d\chi(t). \]

\[ \delta dx_i = \delta(x_i + dx_i - x_i) = \delta(x_i + dx_i) - \delta(x_i) = \epsilon \chi(t + dt) - \epsilon \chi(t) = \epsilon d\chi(t). \]

V. Struik [98] p.135, Exercise 21

Proof. The proof requires a lot of computations. To prevent readers from getting lost, I will divide the proof into several stages so that for each stage readers have a small goal to achieve and a small result to check if they have made any mistakes in this stage.

1. By Struik [98] p.107, (2-6); p.108, (2-9),

\[ \frac{d\mathbf{N}}{dv} = ((\epsilon F - \epsilon G)\mathbf{v} + (\epsilon F - \epsilon G)\Gamma_{12} + \epsilon F - \epsilon G\Gamma_{22})\mathbf{x}_u \]

\[ + \left( \left((\epsilon F - \epsilon G)\mathbf{v} + (\epsilon F - \epsilon G)\Gamma_{12} + \epsilon F - \epsilon G\Gamma_{22}\right)\mathbf{x}_v, \right) \]

\[ + \left( \left((\epsilon F - \epsilon G)\mathbf{v} + \epsilon F - \epsilon G\Gamma_{22}\right)\mathbf{x}_u \right) \]

\[ + \left( \left((\epsilon F - \epsilon G)\mathbf{v} + \epsilon F - \epsilon G\Gamma_{22}\right)\mathbf{x}_v \right). \]

2. By equating the coefficients of \( x_u \) in \( \frac{d\mathbf{N}}{dv} \) and \( \frac{\partial \mathbf{N}}{\partial u} \), we have

\[ (1) \left( \frac{\partial f}{\partial v} - \frac{\partial g}{\partial u} \right) F + \frac{\partial f}{\partial u} G = 0, \]

where

\[ A = gFGEu + gEFGu - gE^2Ev - fG^2E_u + 2fFGE_v - fFEG_v - fEFG_v + eG^2E_u - 2eFGF_v. \]

By equating the coefficients of \( x_v \) in \( \frac{d\mathbf{N}}{dv} \) and \( \frac{\partial \mathbf{N}}{\partial u} \), we have

\[ (2) \left( \frac{\partial e}{\partial v} - \frac{\partial f}{\partial u} \right) E + \frac{\partial f}{\partial u} G = 0, \]

where

\[ B = fFGE_u + 2FEGG_u - 2F^2F_v - gG^2G_v + 2GEGF_u - eEFG_v + eF^2G_u - 2eF^2E_u. \]

3. Prove [the bracket part of (1)] \( \times F + [\text{the bracket part of (2)}] \times G = (\text{the right-hand side of Struik [98] p.111, (3-4)(ii)}) \times (F^2 - EG). \]

4. Prove [the bracket part of (1)] \( \times E + [\text{the bracket part of (2)}] \times F = (\text{the right-hand side of Struik [98] p.111, (3-4)(ii)}) \times (F^2 - EG). \]


VI. The idea of reducing PDEs to ODEs may greatly simplify the above proof [O’neill [74] p.299, Theorem 8.3].

Reducing PDEs to ODEs is an important method in dealing with PDEs in mathematics or physics [See Example 5.13].

Important steps of proving O’neill [74] p.299, Theorem 8.3]:

1. Fix \( p \in M \). For an arbitrary point \( q \in M \), by the connectedness of \( M \), there is a curve \( \alpha \) in \( M \) from \( p \) to \( q \). This step allows us to reduce the above PDEs to ODEs [Struik [98], p.29, (8-6); p.126, l.9–l.9]; O’neill [74] p.117, Lemma 5.2; p.118, l.10, p.121, l.8–l.8].

2. Construct an isometry \( F \) of \( \mathbb{R}^3 \) that carries the initial point \( p \in M \) to the initial point \( \mathbf{p} \in M \) and the initial frame to the corresponding initial frame [O’neill [74] p.299, l.6–l.1.2]].

3. \( F \circ \alpha \) has properties given in O’neill [74] p.300, (A1), (A2)]; \( F \circ \alpha \) has properties given in O’neill [74] p.300, (B1), (B2)].
4. (a). Lemma [O’neill [74, p.121, Theorem 5.7]].

Let \( \alpha, \beta : I \to \mathbb{R}^3 \) be arbitrary curves, and let \( E_1, E_2, E_3 \) be a frame field on \( \alpha; F_1, F_2, F_3 \) be a frame field on \( \beta; F \) be an isometry that carries the frame \( E_i(0) \) at \( \alpha(0) \) to the frame frame \( F_i(0) \) at \( \beta(0) \).

If (1) \( \alpha' \cdot E_3 = \beta' \cdot F_i \),
then \( F \circ \alpha = \bar{\alpha} = \beta \).

(b). The above lemma is an extension of O’neill [74, p.117, Theorem 5.3] [O’neill [74, p.121, l.1–l.2]] because their proofs have the same pattern:

The form of \( f \): O’neill [74, p.118, l.11] \( \iff \) O’neill [74, p.121, l.18];
\( \bar{T} = T \): O’neill [74, p.118, l.8] \( \iff \) O’neill [74, p.121, l.12];
\( \bar{\alpha}' = \beta' \): O’neill [74, p.118, l.8] \( \iff \) O’neill [74, p.121, l.11–l.10–l.10];
O’neill [74, p.117, Lemma 5.2]: O’neill [74, p.118, l.10] \( \iff \) O’neill [74, p.121, l.8].

(c). The identity map carries the frame \( F, E_i \) at \( p \) to the frame \( \bar{E}_i \) at \( \bar{p} \).

\( (F \circ \alpha)' \cdot F, E_i = \alpha' \cdot \bar{E}_i \) [O’neill [74, p.301,(C2)]].

\( (F, E_i)' \cdot F, E_j = \bar{E}_i \cdot \bar{E}_j \) [O’neill [74, p.301,(C1)]].

By Lemma, \( F \circ \alpha = \bar{\alpha} \).

Example 6.105. (To be reader-friendly, a textbook should use common notations and present its proofs concisely)

Blaga [11, p.168, l.14–l.8] use \( r_i, r_{ij} \) instead of the common notations \( x_i, x_{ij} \) given in Kreyszig [61, l.56, l.7]. If a textbook contains strange notations, readers may have to search for the entire book to find where they appear for the first time unless it has a notation index. Thus, readers have to waste a lot of time just for finding the meaning of a notation. Copyright may lead an author to choosing different proofs or notations. In my opinion, it is appropriate to apply copyright to the discussion or interpretation of formulas, but not directly to formulas themselves.

It takes only fifteen lines to prove Kreyszig [61, p.139, Theorem 42.1] by using unit-speed curves, while it takes almost two pages [Blaga [11, p.178, l.13–p.179, l.1–l.7]] to prove the same theorem by using arbitrary-speed curves. There are no new ideas in the latter proof. The use of arbitrary-speed curves in the latter proof complicates each step of the former proof. Thus, in proving a curve theorem, we should parameterize a curve by its arc length \( s \) rather than arbitrary \( t \). The chain rule can easily convert the version of unit-speed curve to the version of arbitrary-speed curve. See Kreyszig [61, p.139, l.12]. In this sense, the entire section of O’neill [74, §II.4] can be omitted except how we interpret the definitions given in O’neill [74, p.66, l.13–l.17].

Example 6.106. (The variation of the main body vs. the variation of its accessories)

Theorem. Suppose \( \kappa > 0 \). A curve \( \beta(s) \) is plane \( \iff \tau = 0 \).

The main body: O’neill [74, p.58, Theorem 3.2].

Its accessories: Struik [98, p.17, (5.5a); (5.5b)].

I. Proof using accessories: Struik [98, p.17, l.1–l.9–l.6; p.18, l.1–l.7].

The proof of Struik [98, p.17, (5.5a); (5.5b)] requires tremendous computations. We need deal with the degenerate case: Struik [98, p.18, l.6–l.7]. From the proof of \( \iff \) part, we do not have any information about the plane’s normal.

II. Proof using the main body: Proof of O’neill [74, p.61, Corollary 3.5].

The Frenet frame field provides the inner structure of the curve. The assumption \( \kappa > 0 \) eliminates the degenerate case. In the proof of \( \iff \) part, the information about the plane’s normal is under control.

Remark. The above theorem is formulated from the conjecture given in O’neill [74, p.61, l.1–l.8–l.7], which is based on the discussion about the Frenet approximation [O’neill [74, p.61, l.10–l.8]]. The Frenet
approximation depends on how we choose the dominated term in each component [O’neill [74] p.61, 1.2–1.3]]. In contrast, the Taylor series approximation given in O’neill [74] p.60, 1.14] has no way to provide any information about the shape of the curve.

**Example 6.107.** (Contact of finite order)

Not a single textbook in differential geometry has a complete discussion in this topic. Struik [98, §1.7] leaves a gap in Struik [98] p.23, 1.–10–1.9]. In part (a) of the proof of Kreyszig [61] pp.48–49, Lemma 14.2], the proof of \( p^{(m+1)}(s_0) \neq 0 \) is incorrect. Thus, a tiny negligence may lead to a big mistake. It easily invites misinterpretations that the same notation \( d \) is used to to express different meanings [Carmo [18] p.158, 1.7–1.8; p.173, 1.5–1.6]]. One cannot even find the important geometric term “contact” either in the index in O’neill [74] or that of Spivak [95]. I plan to divide the discussion of this topic into three parts: I. This part corrects the errors in Kreyszig [61, §14]. II. This part provides the solutions of Carmo [18] pp. 170–171, Exercises 8, 9, 10]. III. This part fills the gap in Struik [98] p.23, 1.–10–1.9].

I. I.

(1). \( \frac{d^{(m+1)}\alpha_j}{ds^{m+1}} \neq \frac{d^{(m+1)}\beta_j}{ds^{m+1}} \) (j = 1, 2, 3) [Kreyszig [61] p.47, 1.–2, (14.2b)] should have been corrected as “\( \exists j \in \{1, 2, 3\} : \frac{d^{(m+1)}\alpha_j}{ds^{m+1}} \neq \frac{d^{(m+1)}\beta_j}{ds^{m+1}} \)."

(2). \( \frac{d^{(m+1)}\alpha_j}{ds^{m+1}} |_{s_0} \neq \frac{d^{(m+1)}\beta_j}{ds^{m+1}} |_{s_0} \) (j = 1, 2, 3) [Kreyszig [61] p.49, 1.–11, (14.5b)] should have been corrected as “\( \exists j \in \{1, 2, 3\} : \frac{d^{(m+1)}\alpha_j}{ds^{m+1}} |_{s_0} \neq \frac{d^{(m+1)}\beta_j}{ds^{m+1}} |_{s_0} \)."

(3). This yields (14.3).

**Proof.** \( p^{(m+1)}(s) = 0 \) cannot occur. Otherwise, by part (b) of the proof of Kreyszig [61] pp.48–49, Lemma 14.2], \( S \) would have contact of order \( \geq m + 1 \) with \( C \) at \( P_0 \), which contradicts the hypothesis. \( \blacksquare \)

Remark. This shows that part (b) of the proof of Kreyszig [61] pp.48–49, Lemma 14.2] should have been put before part (a).

(4). (14.5b) is satisfied for \( j = 3 \) [Kreyszig [61] p.49, 1.–2–1.1].

**Proof.** Let \( p_1(s^3) = G(\beta_1(s^3), \beta_2(s^3), \beta_3(s^3)) \). Then \( p_1(s^3) \equiv 0 \).

If \( \frac{d^{(m+1)}\alpha_j}{ds^{m+1}} |_{s_0} = \frac{d^{(m+1)}\beta_j}{ds^{m+1}} |_{s_0} \), then \( p^{(m+1)}(s_0) = p_1^{(m+1)}(s_0) = 0 \). We would have a contradiction. \( \blacksquare \)

II.


a.

**Proof.** Let \( \textbf{x}_1, \textbf{x}_1 \) be the parametrizations that satisfy the definition of contact.

(i). Since \( f \circ \textbf{x}_1 \circ \textbf{x}_1^{-1} = f = f \circ \bar{\textbf{x}}_1 \circ \bar{\textbf{x}}_1^{-1} \) in \( \textbf{x}_1(U) \cap \bar{\textbf{x}}_1(U) \).

The partial derivatives of order \( \leq 2 \) of \( f \circ \textbf{x} : U \rightarrow \mathbb{R} \) are zero in \( \textbf{x}_1^{-1}(p) \) if and only if the partial derivatives of order \( \leq 2 \) of \( f \circ \bar{\textbf{x}} : U \rightarrow \mathbb{R} \) are zero in \( \bar{\textbf{x}}_1^{-1}(p) \).

(ii). Let \( \textbf{x} = \textbf{x}_1 \circ \textbf{h} \).

\[
\frac{df}{du} = \sum_{i=1}^{3} \frac{df}{dx_{i1}} \frac{dh_i}{du}, \quad \frac{df}{dx_{i1}} = \frac{df}{\partial \bar{x}_i} \frac{\partial \bar{x}_i}{\partial u} \text{ (by (i))}. \]

\( \blacksquare \)

c.
Proof. Let \( x = (x, y, f(x, y)) \) and 
\[
\bar{s} = (x, y, \frac{1}{2}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy})).
\]
\( x = (1, 0, f_x) = (1, 0, 0) \) [since \( x \) must be on the tangent plane \( z = 0 \)]
\[
= \bar{s}_y.
\]
d.

Proof. \( \bar{f}(x, y) = ax^2 + 2bxy + cy^2 + dx + ey \) [Carmo \[18\] p. 485, l.–20] follows from Bell \[6\] p.122, §79, (1).
Let \( P \) be the paraboloid that has contact of order \( \geq 2 \) with \( S : z = f(x, y) \) at \( P \).
P\( O \) be the osculating paraboloid of \( S \) at \( p \).
By the definition of contact, \( P \) has contact of order \( \geq 2 \) with \( P_O \).
By \( b, d = c = 0, a = \frac{1}{2}f_{xx}, b = f_{xy}, c = \frac{1}{2}f_{yy} \).

Part e follows from Carmo \[18\] p. 163, l.16–1.17.
f. \( U \times \mathbb{R}^3 \ni (u, v) \)
\[
(u, v) \mapsto \phi(y_1, y_2, y_3).
\]
\[
\frac{\partial \phi}{\partial u} = \sum_{i=1}^3 \frac{\partial \phi}{\partial y_i} \frac{\partial y_i}{\partial u} = \sum_{i=1}^3 \frac{\partial \phi}{\partial y_i} \frac{\partial y_i}{\partial u} = \frac{\partial \phi}{\partial u}.
\]
g. \( S, \bar{S} \) have contact of order \( \geq 2 \) if and only if \( \lim_{r \to 0} \frac{r}{d} = 0 \).

Proof. \( \Rightarrow \): Let \( xy \)-plane be the common tangent plane of \( S, \bar{S} \) at \( p \) as in Carmo \[18\] p. 170, Exercise 8.c].
By Carmo \[18\] p. 159, l.9–l.6,
\[
x(u, v) = x(0, 0) + x_u(0, 0)u + x_v(0, 0)v + \frac{1}{2}(x_{uu}(0, 0)u^2 + 2x_{uv}(0, 0)uv + x_{vv}(0, 0)v^2) + R,
\]
where \( \lim_{(u, v) \to (0, 0)} \frac{R}{u^2 + v^2} = 0 \);
\[
\bar{x}(u, v) = \bar{x}(0, 0) + \bar{x}_u(0, 0)u + \bar{x}_v(0, 0)v + \frac{1}{2}((\bar{x}_{uu}(0, 0)u^2 + 2\bar{x}_{uv}(0, 0)uv + \bar{x}_{vv}(0, 0)v^2) + R,
\]
where \( \lim_{(u, v) \to (0, 0)} \frac{R}{u^2 + v^2} = 0 \).
\[
d = ||(x(u, v) - \bar{x}(u, v)) \cdot N|| = ||[R - \bar{R}] \cdot N|| \to 0 \) as \( r^2 = u^2 + v^2 \to 0 \).
\( \Leftarrow \): Let \( x(u, v) \) be the parametrization of \( S : z = f(x, y) \) and \( \bar{x}(u, v) \) be the parametrization of \( \bar{S} : z = \bar{f}(x, y) \).
By a method similar to the one used to prove Blaga \[11\] p.40, Theorem 1.6.2],
\[
(x_u - \bar{x}_u) \cdot N = 0, \text{ where } N = (0, 0, 1). \text{ Thus, } f_x = \bar{f}_x. \text{ Therefore, } x_u = \bar{x}_u \text{ at } (u, v) = (0, 0).
\]

(2). Carmo \[18\] p.171, Exercise 9]
a. Similar to Carmo \[18\] p. 170, Exercise 8.f].
b. Two curves have contact of order \( \geq 1 \) if and only if they have the same tangent vector at \( p \) [by definition of contact].

(3). Carmo \[18\] p.171, Exercise 10]
b. Blaga \[11\] p.66, l.8–1.13].
c. Blaga \[11\] p.68, l.1–1.14].

III.
(1) \( AD = F(x_1, y_1, z_1) = o(x_1^2 + y_1^2) \) [Struik \[98\] p.23, l.–10–l–.9]].
Proof. A. Case \( m = 2 \): Let \( F(x,y,z) = z - f(x,y) \).
(i). \( AD = O(x^2_1 + y^2_1) \) [Struik [98] p.23, 1.19].
(ii). \( f(x_1,y_1, z_1) = \left[ \frac{1}{2}(ex^2_1 + 2fx_1y_1 + gy^2_1) + o(x^2_1 + y^2_1) \right] - z_1 \) [Carmo [18] p. 158, l.4–1.11]
\( = \left[ \frac{1}{2}(ex^2_1 + 2fx_1y_1 + gy^2_1) + o(x^2_1 + y^2_1) \right] = -R + o(x^2_1 + y^2_1) \) [Carmo [18] p. 158, l.2–1.11]
\( = o(x^2_1 + y^2_1) \) [Carmo [18] p. 158, l.1–1.11].
(iii). \( AD - F(x_1,y_1, z_1) \)
\( = AD - [z_1 - f(x_1,y_1)] \)
\( = o(x^2_1 + y^2_1) \) (by (i), (ii)).
B. The proof of the general case is similar.

(2). \( f(u_1) \) is of the order of \( AD \) [Struik [98] p.24, l.5; p.42, Exercise 4]].

Proof. The following proves the case \( \Sigma_2 : (x - a)^2 - r^2 = 0 \).
Without loss of generality we may assume \( a = 0 \).
Let \( D = ry_1, A = x_1 = (r + \varepsilon)y_1 \), where \( |y_1| = 1 \).
\( x_1^2 - r^2 = 2r\varepsilon + \varepsilon^2 = 2rAD + o(\varepsilon) \).
Consequently, \( AD = O(x^2_1 - r^2) \).

(3). The necessary and sufficient conditions that the surface has a contact of order \( n \) at \( P \) with the curve are that at \( P \) the relations hold:
\( f(u) = f'(u) = f''(u) = \cdots = f^{(n)}(u) = 0; f^{(n+1)}(u) \neq 0 \) (7-4) [Struik [98] p.24, l.1–1.8]]

Proof. \( \lim_{h \to 0} \frac{AD_{AP}}{AP} = 0 \iff \lim_{h \to 0} f(u_1) = 0 \iff f(u_0) = 0. \)
\( \lim_{h \to 0} \frac{AD_{AP}}{AP} = 0 \iff \lim_{h \to 0} \frac{f(u_1)}{h} = 0 \iff f'(u_0) = 0. \)
\( \vdots \)
\( \lim_{h \to 0} \frac{AD_{AP}}{AP} = 0 \iff \lim_{h \to 0} \frac{f^{(n)}(u_1)}{h^n} = 0 \iff f^{(n)}(u_0) = 0, \) where \( f(u_1) = \frac{h^n}{n!} f^{(n)}(u_0) + o(h^n). \)
\( \lim_{h \to 0} \frac{AD_{AP}}{AP} = L(\text{finite}) \neq 0 \iff \lim_{h \to 0} \frac{f(u_1)}{h^n} = L \neq 0 \iff f^{(n+1)}(u_0) \neq 0, \)
where \( f(u_1) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(u_0) + o(h^{n+1}). \)


Proof. Let \( F_1(x,y,z) = 0 \) be the equation of \( S_1 \) and \( f_1(u) = F_1(\alpha_1(u), \alpha_2(u), \alpha_3(u)) \).
By Struik [98] p.10, 1. – 16–1. – 1; p.24, 1. – 7–1. – 1,
\( f_1(u) = f'_1(u) = f''_1(u) = \cdots = f^{(n)}(u) = 0 \) means that \( \alpha \) and \( S_1 \) have at least \( m + 1 \) consecutive forward points in common at \( P \).

Example 6.108. (The Lie–Darboux method of resolving an illusive contradiction by substituting true solutions into the required conditions) [Blaga [11], §3.1; Struik [98] §1-10]
Struik [98] p.38, 1.18–1.19] considers both the compatibility between \( c_3d_3 = -1 \) and \( (c_3 = \infty, d_3 = 0) \)
and that between \( c_3d_3 = 1 \) and \( (c_3 = \infty, d_3 = 0) \). I wonder why it does not consider the compatibility between
$c_3d_3 = -1$ and $c_3d_3 = 1$ because $\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 0$ and $\alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 = 0$ have to be simultaneously established. Perhaps Struik fails to know how to resolve this contradiction. Obviously, he fails to fully understand Lie–Darboux’s work.


II. The proof of Blaga [111, p.72, Theorem 1.14.3] is based on the existence and uniqueness of solutions to ODEs. The proof of the latter theorem is based on the method of successive approximations, so it uses an infinite number of quadratures. In contrast, the solutions $(w_i, z_i)$ of the Riccati equation [Struik [98, p.38, l.–1]] are determined in the Lie–Darboux theorem [Struik [98, p.38, l.–4–p.39, l.3]] can be put in the form as in Struik [98, p.38, l.–1] using 0 [1, 2 resp.] quadratures if 3 [2, 1 resp.] of its solutions are known [Eisenhart [31, p.26, l.1–l.1–l.1]].

III. Struik [98, p.37, l.–1–p.38, l.19] is at best a simple, efficient tool for testing solution candidates and provides crude information for our decision about selection. Its advantage lies in the fact that complicated choices can be removed in advance. However, we have to impose some conditions like $\infty - \infty = 0$ and $\infty - \infty = 0$. These restrictions might be the reason that leads to contradictory conclusions: $c_3d_3 = -1$ and $c_3d_3 = 1$. It may be possible to obtain the choices given in Struik [98, p.38, l.16] by other means without imposing the above restrictions. Once the restrictions are removed, the above contradiction may disappear. Consequently, until $(\alpha_i, \beta_i, \gamma_i)$ are definitely determined, it is simply not a proper timing to discuss compatibility. This is the first mistake that Struik commits. We may also treat this problem from the viewpoint of calculation. Although the calculation in checking if the given $(c_i, d_i)$’s [Struik [98, p.38, l.16]] satisfy the equations in Struik [98, p.38, l.10–l.13] does not involve $f_i$, we still have to face the difficulty of calculating $\infty \cdot 0$. Thus, at this stage the operations involving $\infty \cdot 0$ fail to have definite value. It is not the mature time to discuss compatibility. Therefore, it is better to find $(\alpha_i, \beta_i, \gamma_i)$ first and then check if they satisfy the equations given in Struik [98, p.37, l.–6]. Since this calculation involves only definite values, the compatibility problem will not occur. Struik fails to complete this type of verification. This is the second mistake that he commits.

In contrast, Eisenhart [31, p.27, l.–9–p.28, l.8; p.28, l.–4–l.3] provide the correct argument.

IV. $\alpha_i$’s are given in Struik [98, p.38, l.–1–l.6]. Similarly,

$\beta_i = i\frac{(f^2_i - f^2_j) + (f^2_j - f^2_i)}{f^3_i f^3_j - f^3_j f^3_i}, \beta_2 = \frac{f^2_j + f^2_j + f^2_j}{f^3_j f^3_j - f^3_j f^3_i}, \beta_3 = \frac{f^2_j + f^2_j}{f^3_j f^3_j - f^3_j f^3_i}, \gamma_1 = -\frac{f^2_j + f^2_j}{f^3_j f^3_j - f^3_j f^3_i}, \gamma_2 = -\frac{f^2_j + f^2_j}{f^3_j f^3_j - f^3_j f^3_i}, \gamma_3 = \frac{f^2_j + f^2_j}{f^3_j f^3_j - f^3_j f^3_i}$. They satisfy the equations given in Struik [98, p.37, l.–6].

**Example 6.109.** (Developables)

I. (Situation control) Intrinsically, the theorem given in Struik [98, p.67, l.–16–l.–12] is a local theorem. If one were to present it as a global theorem, it would not add much significance to its theme. The condition that $(a(u), a'(u), a''(u))|_{u=u_0} \neq 0$ is all we need for this theorem. By continuity, there is a neighborhood of $u = u_0$ for which $(a(u), a'(u), a''(u)) \neq 0$. This theorem essentially discusses this neighborhood alone. Thus, characteristic points always exist in this neighborhood. We may exclude the pencil case, but we do not need the general hypothesis that the family of planes are not all parallel because Struik fails to discuss the case $(a(u), a'(u), a''(u)) \neq 0$ except $u = u_0$.

II. (Making the concept of consecutiveness rigorous by using Rolle’s theorem)

The treatment given in Struik [98, p.66, l.7–l.19] makes the following statement rigorous:

A characteristic line is the intersection of two consecutive planes.

The treatment given in Struik [98, p.66, l.7–l.27] makes the following statement rigorous:

A characteristic point is the intersection of three consecutive planes.

Following the above treatments, we can make the definitions of envelope [Weatherburn [110, vol. 1, p.40, l.–6; p.48, l.–10]], edge of regression [Weatherburn [110, vol. 1, p.42, l.3]], and developable surfaces
III. (Duality between space curves and developables)

Struik [98, p.72, l.10–l.10] indicates a duality between space curves and developables. This duality does not mean much because there are no strict rules that we can follow to find the dual of a statement. It just comes from the attempt to associate the statements in the proof of Struik [98, p.64, l.1–l.13] with the corresponding statements in the proof of Struik [98, p.71, l.1–l.1]. The relationship between the two proofs has a somewhat dual taste.

**Example 6.110.** (The remarkable theorem considered by Gauss)


II. Weatherburn [110, vol. 1, p.93, (6)].

**Proof.** (a). The right-hand side = \( \frac{1}{2} \left( 2F_{12} - E_{22} - G_{11} \right) + A \), where

\[
A = \frac{1}{2} H \left[ \frac{(E H F_1 - F (E H)_1) E_2 + H_1 H_2 G_1 - 2 H_1 F_1 + H_2 E_2 - \frac{E H F_2 - F (E H)_2}{(E H)^2} E_1}{} \right] \\
= \frac{1}{4 H} \left[ 2G_1 E_2 + F_1 E_2 + F_2 G_1 - F_2 E_2 + G_1 E_2 + 2H_1 F_1 - 4H_2 F_1 + 2H_2 E_2 \right] \\
= \frac{B}{4 H},
\]

where

\[
B = \left( E G - F^2 \right) \left( -2E_1 F_2 + F_1 E_2 + F_2 E_1 + G_1 E_2 - F_2 G_1 - 2E_1 F_2 + 4F_1 F_2 + G_2 + 2E_2 G_2 - 2F_2 G_2 \right)
\]

(b). \((m^2 - l n) E = \frac{1}{4 H^2} \left[ E G \left( 2E^2 G - 2E F_2 G_1 + EF_2 G_1 - 2E G_2 E_2 F_2 + 4E F G_1 + 2F_G E_2 \right) \right].
\]

(c). \(2m \mu - (l v + \lambda n) F = \frac{1}{4 H^2} \left[ 2E F_2 G_1 - 2E G^2 E_2 + 3E_2 E_2 - 2E G_2 E_2 F_2 + 4E F G_1 + 2F_G E_2 \right].
\]

(d). \((\mu^2 - \lambda v) G = 2E G_2^2 + 2E F_2 G_1 + 2E G_2 E_2 - 2E G_2 E_2 F_2 + 2E G_1 G_2 + 2E G_2 E_2 + 4E F G_1 + 2F_G E_2 \).

**Example 6.111.** (The key to the proof of the equivalence of Codazzi equations and the compatibility conditions for Weingarten equations lies in the insight into inner structures rather than long calculations)

\( N_{12} = N_{21} \Leftrightarrow \) Struik [98, p.111, (3–4)].

**Proof.** I. By Struik [98, p.108, (2–9)].
\[ \mathbf{N}_{12} = \begin{pmatrix} F - eG \\ G - F \end{pmatrix} \mathbf{x}_{12} + \frac{(E - F^2)(f_2 F + f F_2 - e_2 G - e G) - (F - e G)(E_2 G + E G_2 - 2 F F_2)}{(E - F^2)^2} \mathbf{x}_1 \\
+ \frac{e F - e G}{E - F^2} \mathbf{x}_{22} + \frac{(E - F^2)(e_2 F + e F_2 - f_2 E - f E_2) - (e F - e G)(E_2 G + E G_2 - 2 F F_2)}{(E - F^2)^2} \mathbf{x}_2. \]

\[ \mathbf{N}_{21} = \begin{pmatrix} g_2 F - G \\ E - F \end{pmatrix} \mathbf{x}_{11} + \frac{(E - F^2)(g_2 F + g_2 F_1 - f_2 G - f G_1) - (g F - G)(F G_1 + E G_2 - 2 F F_2)}{(E - F^2)^2} \mathbf{x}_1 \\
+ \frac{f F - e G}{E - F^2} \mathbf{x}_{21} + \frac{(E - F^2)(f_2 F + f_2 F_1 - g_2 E - g E_1) - (f F - e G)(E_1 G + E G_2 - 2 F F_2)}{(E - F^2)^2} \mathbf{x}_2. \]

II. By Struik [98, p.107, (2-4)],
(the coefficient of \( x_1 \) in \( \mathbf{N}_{12} \) = the coefficient of \( x_1 \) in \( \mathbf{N}_{21} \)) \( \iff \)
\[
\frac{g_2 F - G}{E - F^2} \mathbf{\Gamma}_1^2 + \frac{e F - e G}{E - F^2} \mathbf{\Gamma}_2^2 - \frac{g F - f G}{E - F^2} \mathbf{\Gamma}_1^1 = \frac{(E - F^2)(g_2 F + g_2 F_1 - f_2 G - f G_1) - (g F - G)(E_1 G + E G_2 - 2 F F_2)}{(E - F^2)^2}.
\]

III. By Struik [98, p.107, (2-7)],
\[
\frac{g E - e G}{E - F^2} \mathbf{\Gamma}_1^2 + \frac{e F - f E}{E - F^2} \mathbf{\Gamma}_2^2 - \frac{g F - f G}{E - F^2} \mathbf{\Gamma}_1^1 = \frac{\Gamma_A}{(E - F^2)^2},
\]
where
\[
A = g E G F_2 - g E F G_1 - e G^2 E_2 + e F G G_1 + 2 e F G F_2 - e F G G_1 - e F^2 G_2 - 2 f E G F_2 + f E G G_1 + f E F G_2 - g F G E_1 + 2 g F^2 F_1 - g F^2 E_2 + f G^2 E_1 - 2 f F G F_1 + f E G E_2.
\]

IV. Let \( B = (E - F^2)(f_2 F + f_2 F_2 - e_2 G - e G_2) - (f F - e G)(E_2 G + E G_2 - 2 F F_2) \).

Then \( B = E F G G_1 + g E G F_1 - E^2 G F_1 - f E G G_1 \).

V. Let \( C = (E - F^2)(f_2 F + f_2 F_2 - e_2 G - e G_2) - (f F - e G)(E G_1 + E G_2 - 2 F F_2) \).

Then \( -C = -E F G f_2 - f E G F_2 + E^2 G_2 + e E G G_2 + F^2 f_2 + f F G F_2 - F^2 G G_2 - e F G G_2 + e F G F_2 - 2 F^2 F_2 - e G G G_2 + 2 e F G F_2. \)

VI. Let \( \text{Red} = \text{The sum of all red terms in } B \text{ and } -C. \)

Then \( \text{Red} = -F(E - F^2)(f_2 - g_2) + G(E - F^2)(f_2 - g_2) \).

Then \( = -F(E - F^2)(f_2 - g_2) + G(E - F^2)(f_2 - g_2) \).

Then \( = -(E - F^2)[-\Gamma_2^1 e F + \Gamma_1^2 f - \Gamma_2^2 f F + \Gamma_2^1 g F + \Gamma_1^2 e G - \Gamma_1^1 f G + \Gamma_2^1 f G - \Gamma_1^2 g G]. \)

By Struik [98, p.107, (2-7)],

2 \( \text{Red} \)

Then \( = f F G E_1 + e G^2 E_2 - f F^2 G_1 - e F G G_1 - 2 F^2 E_1 + 2 f F G F_1 - f F G G_2 + f E G G_1 - 2 g E G F_1 + g E G E_2 + g F G E_1 - 2 F G F_2 + e F G G_1 + e F^2 G_2 - f E F G_2 + 2 F F F_2 - f F^2 G_2. \)

VII. Let \( \text{Black} = \text{The sum of all black terms in } B \text{ and } -C. \)

Then \( \text{Black} = \frac{2 g E G F_1 - 2 F E G G_1 - 2 F^2 F_2 + 2 F F^2 G_1 - 2 g F G E_1 - 4 F F G_1 + 4 F^2 F_1}{2 F G^2 E_1 + 2 F E G F_2 - F^2 G G_1 - 2 F F^2 G_2 - 2 F F G E_2 + 2 F F E G_2 - 4 F^2 F_2 - 2 E G^2 E_2 - 2 e E G G_2 + 2 e F G F_2}. \)
VIII. Let $S = e_2 - f_1$ and $T = f_2 - g_1$. Then $\text{Red} = -FT + GS + \cdots$.

By substituting $\Gamma_1 e = (\Gamma_1^1 - \Gamma_1^2) f - \Gamma_1^2 g$ [resp. $\Gamma_2 e = (\Gamma_2^1 - \Gamma_2^2) f - \Gamma_2^2 g$] into $S$ [resp. $T$] of $2(B - C) = 2$ Red + 2 Black, we will get $A$. Thus,

$-FT + GS = a$, where $a$ can be expressed in terms of $[E, F, G; E_1, F_1, G_1; E_2, F_2, G_2, e, f, g]$.  

II'. By Struik [98, p.107, (2-4)],  

(the coefficient of $x_2$ in $N_{12}$ = the coefficient of $x_2$ in $N_{21}$) $\Leftrightarrow$

$\frac{gE - eG}{E G - F^2} \Gamma_1^{12} + \frac{eF - fE}{F G - F^2} \Gamma_2^{12} = \frac{E G - E^2}{E G - F^2} \Gamma_1^{11}$

$= (E G - F^2)(f_1 F + f_1 f - g_1 E - g_1 E_1) - (f F - g E)(E_1 G + E G_1 - 2 F F_1)$.

III'. By Struik [98, p.107, (2-7)],

$\frac{gE - eG}{E G - F^2} \Gamma_1^{12} + \frac{eF - fE}{F G - F^2} \Gamma_2^{12} = \frac{E G - E^2}{E G - F^2} \Gamma_1^{11}$

$= \frac{A'}{F G - F^2}$, where

$A' = gE^2 G_1 - gE E F_2 - eEG G_1 + eF G E_2$

$+ eE F G_2 - 2 eF^2 F_2 + eF^2 G_1 - fE^2 G_2 + 2 fE F F_2 - fE F G_1$

$- 2 E F F_1 + eE F G_2 + gF^2 E_1 + 2 E F G F_1 - fE G E_2 - fE F G_1$.

IV'. Let $B' = (E G - F^2)(f_1 F + f_1 f - g_1 E - g_1 E_1) - (f F - g E)(E_1 G + E G_1 - 2 F F_1)$. Then

$B' = E F G F_1 + fE G F_1 - E^2 G G_1 - gE G E_1$

$- F^2 f_1 - f F^2 F_1 + E F^2 G_1 + gF^2 E_1$

$- fE F G_1 - fE F G_1 + 2 F F^2 F_1$

$+ gE G E_1 + gE^2 G_1 - 2 gE F G_1$.

V'. Let $C' = (E G - F^2)(e_2 F + e_2 F - f_2 E - f_2 E_2) - (e F - f E)(E_2 G + E G_2 - 2 F F_2)$. Then

$-C' = -E F G E_2 - eE F G_2 + E^2 G F_2 + fE G E_2$

$+ F^2 e_2 + eF^2 F_2 - E^2 F_2 - fE^2 E_2$

$+ E F G E_2 + E F G F_2 - 2 E F^2 F_2$

$- fE G E_2 - fE^2 G_2 + 2 fE F F_2$.

VI'. Let $\text{Red}'$ = the sum of all red terms in $B'$ and $-C'$. Then

$\text{Red}' = F (E G - F^2)(e_2 F + e_2 F_1 + eF G F_1 - E G - F^2)(f_2 - g_1)$

$= (E G - F^2)(f_1 F + f_1 f - g_1 E - g_1 E_1) - (f F - g E)(E_1 G + E G_1 - 2 F F_1)$

(by Struik [98] p.111, (3-4)).

By Struik [98, p.107, (2-7)],

$2 \text{Red}'$

$= fE F G_1 - 2 f F^2 F_1 + f F^2 E_1 + eF G E_2 - gE G E_1 - 2 F F F_1 - eE F G_2 - eF^2 G_1$

$- eE F G_1 + eF^2 G_2 - eF F G_2 + 2 eE G F_2 - eE G E_2 - eF G E_2 - fE^2 G_2 - 2 F F F_2 - fE F G_1$.

VII'. Let Black' = the sum of all black terms in $B'$ and $-C'$. Then

$2 \text{Black}' = 2 fE G F_1 - 2 gE G E_1 - 2 F F^2 F_1 + 2 gF^2 E_1 - 2 F F G E_1 - 2 F F G F_1 + 4 f F F^2 F_1 + 2 gE G E_1 + 2 gE G G_1 - 4 gE F F_1$

$- 2 eE F G_2 + 2 eE G E_2 + 2 eE F^2 F_2 - 2 eF^2 F_1 + 2 E F G E_2 + 2 eE G F_2 - 4 eF G E_2 - 2 fF G E_2 - 2 fF^2 G_2 + 4 f F F_2$.

Then

$-FS + ET = a'$, where $a'$ can be expressed in terms of $[E, F, G; E_1, F_1, G_1; E_2, F_2, G_2, e, f, g]$.

By I-VIII and II'-VIII', Struik [98, p.111, (3-4)] $\Rightarrow N_{12} = N_{21}$.

By VIII and VIII',
\[
\begin{align*}
G S - F T &= a \\
-F S + E T &= d'
\end{align*}
\]
has a unique solution \((S,T)\) because \(\begin{vmatrix} G & -F \\ -F & E \end{vmatrix} \neq 0\). Consequently, \(N_{12} = N_{21} \Rightarrow \text{Struik} \ [98, \text{p.}111, (3-4)]\). For another proof, see Part V of Example 6.104.

Remark. The Gauss characteristic equation [Weatherburn [110, vol. 1, p.93, (5)] is equivalent to each of the four formulas given in Weatherburn [110, vol. 1, p.96, l.7–l.4] [Blaga [11, p.165, (4.17.14); p.168, l.14]].

To prove that the Gauss characteristic equation implies each of the four formulas given in Weatherburn [110, vol. 1, p.96, l.7–l.4], we have to compare the terms on the left-hand side with the terms on the right-hand side of the formula. For the left-hand side, we have to figure out the terms of the numerator in
\[
K = LN - M^2 E F - G^2.
\]
There are 46 terms not involving second derivatives. However, if we merge the like terms, the 46 terms will be reduced to 22 terms. This will save a lot of comparisons among terms. Similarly, if we merge the like terms on the the right-hand side first, it will also save a lot of comparisons among terms. For example, there are 40 terms not involving second derivatives on the right-hand side of the formula given in Weatherburn [110, vol. 1, p.96, l.5]. If we merge the like terms first, the 40 terms will be reduced to 11 terms. If we multiply these 11 terms with \(E F - G^2\), the denominator of \(K\), we will have 22 terms, which matches the number of terms on the left-hand side. To prove an equality with \(n\) unlike terms on each side, there are only \(\frac{n^2}{2}(n+1)\) comparisons to be checked.

Example 6.112. (Following intuition is the best proof method)

Following intuition is the best proof method. A proof using unnecessary objects may obscure the theorem’s theme and confuse readers. Let us prove \(k = k_e\) [Kreyszig [61, p.138, (42.3)(b)] without using the cylinder given in Kreyszig [61, p.138, l.12] or Struik [98, p.129, l.10; l.15; l.18, l.19].

\[
\begin{align*}
\text{Proof.} \quad k &= k_{n} + k_{e} \quad \text{[Kreyszig [61, p.138, (42.2)]]} \\
&= k_{n} + \frac{dx}{dt} \quad \text{[Kreyszig [61, p.32, (10.5)(i)]]} \\
&= k_{n} + k_{e} \quad \text{[since } C \text{ and } C^{*} \text{ have two common consecutive points at } P] \\
&= k_{n} + k_{e}^{*} \quad \text{Consequently,} \\
k_{e} &= k_{e}^{*} = k_{e} \quad \text{[Kreyszig [61, p.137, l.5–l.1].]} \\
\end{align*}
\]

Example 6.113. (Direct and intuitive definitions of differential of a function on a curve or a surface)

The definition based on tensors on a manifold given in Spivak [95, vol. 1, p.286, l.4] is abstract, complicated and far from the original. The setting of O’Neill [74, p.19, Lemma 4.6] fails to match our needs exactly because the domain of \(f\) should have been the range of \(\alpha\). If we were to correct the setting, it would be the perfect definition for differential of a function on a curve. O’Neill [74] gives a long process of preparing for the definition of differential of a function on a surface [O’Neill [74, chap.I; chap. II, §§3-4; chap. IV, §§3-4; chap. VI, §§1-2, §§4-6]], but it has never provided an explicit definition. I have never seen the formula \(df = f_{u}du + f_{v}dv\) [the surface version, O’Neill [74, p.277, l.16)] in O’Neill [74, chap. IV]. Furthermore, the process of preparation is too long for the definition to be simple, clear, direct, and intuitive. Some parts of it are unnecessary and useless.

The differential of a function on a surface is a generalization of a derivative of a function on the real line [By intuition (straight line \(\rightarrow\) curve), we may immediately obtain the direct and natural extension: O’Neill [74, p.11, Definition 3.1 \(\rightarrow\) p.149, Definition 3.10]]. The latter refers to the differentiation along the positive direction of a straight line [O’Neill [74, p.11, Definition 3.1]], while the former refers to the differentiation along the direction of unit tangent vector (there are a lot of directions on a tangent plane to choose) at each point of a curve. Consequently, for the former, we must introduce the concept of differential \(df\) (or
differential form). Then we may easily extend the definition of $df$ given in O’neill [74, p.23, Definition 5.3] to a function on a surface. After all the relationship between the differential form $df$ and $\alpha'(t)[f]$ is nothing but one between a function and its function values. The definition of directional directive of a function on a surface is given in O’neill [74, p.149, Definition 3.10]. Applications: O’neill [74, p.150, l.2; p.156, l.1; p.397, l.7].

Remark. O’neill [74, p.321, l.9] shows that the covariant derivative $\nabla V W$ of a geometric surface is only required to satisfy the connection equations [O’neill [74, p.318, l.19–l.9]] and is not necessarily to be real. Thus, the naming of covariant derivative of a geometric surface is symbolic in a sense.

Example 6.114. (Parallel postulate [O’neill [74, p.335, l.3–p.336, l.4]])

Before Riemann, there had been many mathematicians who attempted to deduce the parallel postulate in $E^2$ but to no avail. Let us see how Riemann deals with this problem. We pay special attention to where he looks for counterexamples and how he obtains the answer.

The place that he looks for counterexamples: geometric surfaces [O’neill [74, p.305, Definition 1.2]].

His solving method: $E^2$ is a special case of geometric surface. A straight line corresponds to a geodesic in the general case. Riemann tries to construct geometric surfaces such that through a point outside a geodesic there are no or an infinite number of geodesics not intersecting with the given geodesic.

Case of no geodesics: spheres.

Case of an infinite number of geodesics: The hyperbolic plane $H$ [O’neill [74, p.336, Fig. 7.11]].

Remark 1. The sum of the angles in the triangle $\triangle NQL$ in p.23, Fig. 3.1 of [I. R. Kenyon, General Relativity, Oxford: Oxford University press, 1990] is greater than $\pi$. If the vertices of the triangle whose sides are geodesics are on the circle $x^2 + y^2 = 4$, then the sum of angles of the triangle in O’neill [74, p.336, Fig. 7.11] is 0.


Example 6.115. (How we round off a corner of a curve)

One can “round off” this corner, obtaining a curve segment $\gamma$ from $p$ to $r$ which is only slightly longer than $\alpha$ and $\beta$ [O’neill [74, p.347, l.12–l.15]].

Proof. Let $\gamma_1 = \alpha + \beta$ [O’neill [74, p.347, FIG. 7.21]].

We may assume that $[\gamma_1 \in C^\omega((\infty,0) \cup (0,\infty)), \gamma_1(0) = q; \sup_{t_1,t_2 \in [-\delta,\delta]} \rho(\gamma_1(t_1),\gamma_1(t_2)) < \delta]$.

We want to modify the values of $\gamma_1$ in $[-\delta,\delta]$ so that the resulting curve $\gamma$ satisfies $\gamma \in C^\omega(\mathbb{R})$.

Let $I$ be defined as in Spivak [95] vol. 1, p.43, Fig. 4] and $f : \mathbb{R} \to [0,1]$ with

$$f(t) = \begin{cases} -t/(2t + 2\delta) + 1 & \text{if } t \leq 0 \\ f(-t) & \text{if } t > 0 \end{cases}.$$ Then $f \in C^\omega(\mathbb{R})$, $f \equiv 0$ on $[-\delta/2,\delta/2]$, and $f \equiv 1$ in $\mathbb{R} \setminus [-\delta,\delta]$.

Let $\gamma = f \gamma_1$.

Example 6.116. (An incorrect definition leads to an incorrect proof)

When an author fails to make readers understand his or her proof of a theorem, it either means that the proof is incorrect or means that the author fails to grasp the key idea of the proof. Such a “proof” wastes not only the author’s time, but also the readers’ time.

O’neill [74] p.184, Definition 8.4] is incorrect because a differentiable manifold structure is required to
be an equivalent class of atlases [Arnold [4, p.290, Definition 5]) or the maximal atlas [Spivak [95, vol.1, p. 38, l.1–11]) rather than a single atlas. Otherwise, the charts (or patches) in the maximal atlas on the differentiable manifold would have the inconsistency problem.

O’neill [74, p.183, l.13–l.17] is incorrect because it uses small patches [O’neill [74, p.184, l.7–l.8]] alone to avoid checking the consistency between a small patch and a large patch. This consistency must be checked according to the definition of a differentiable manifold [Arnold [4, p.288, l.1–l.11]]). The limitation of using small patches will fail to produce not only the required differentiable manifold structure, but also the required quotient topology.

A correct proof to check consistency among charts in the maximal atlas is given in Arnold [4, p.292, l.12–l.1]. The boundless talk given in Spivak [95, vol.1, p. 13, l.5–p.14, l.1; p.14, the first figure] fails to point out the key idea given in Arnold [4, p.292, l.12–l.1], so it must be nothing but an incomplete version of the proof given in O’neill [74, p.183, l.13–l.17]. Spivak might vindicate himself by saying, “I have used Spivak [95, vol.1, p. 38, Lemma 1] to prove the existence of the maximal atlas containing a certain atlas”. However, it seems that Spivak’s solution to a problem is \( \{x | x \text{ is a solution to that problem}\} \). His proof neither makes any progress toward the goal of constructing a solution, nor provides any effective algorithm to test a candidate if it is a solution. Thus, Spivak mistakes a problem itself for its final solution. Other people prove the existence of maximal atlas using the axiom of choice. The existence in the axiom of choice is assertive, so the existence of maximal atlas produced by such proof is also assertive. The purpose of the theory of axiom of choice is to see what consequential results would be if we were to consider it true. Its advantage: If we can prove the validity of the axiom of choice for a special case, then all its consequential results will be true for that special case. However, before we prove that the axiom of choice for the special case, the above consequential results should not be treated as true theorems. In contrast, Arnold [4, p.292, Example 3; p.291, Fig. 237] proves the consistency of any two of three big charts. It shows how to remove the obstacles of the most impossible case for consistency. Once their consistency problem is solved, to solve the consistency problem for any other two charts would be similar and easier. Consequently, this existence of maximal atlas is constructive.

Remark 1. The last paragraph of §Maximal smooth atlases in https://en.wikipedia.org/wiki/Smooth_structure says, “In general, computations with the maximal atlas of a manifold are rather unwieldy. For most applications, it suffices to choose a smaller atlas.” These claims and similar ones in many textbooks are confusing because they make readers under the impression that any atlas can be used to represent the maximal atlas containing that atlas. This is not true because the consistency between any chart in the maximal atlas and the charts in the given atlas still needs to be checked. However, each of the three atlases given in Arnold [4] pp. 291–292, §33.3] can represent the maximal atlas containing it because the domain of each of its charts cannot be extended further so that the most impossible cases for consistency among the charts in the maximal atlas would be the cases for determining if the charts in the given atlas are consistent. Remark 2. The concept of atlas is useful in differential geometry because the charts in it are consistent. The differentiable manifold structure is defined as an equivalent class of atlases [Arnold [4, p.290, Definition 5]) or the maximal atlas [Spivak [95, vol.1, p. 38, l.1–11]). The drawback of the former definition is that we have to find an effective algorithm to determine if two charts are consistent before we can determine if two atlases are equivalent. Thus, the former definition may easily make us forget to check the consistency problem. The latter definition may contain too many extra charts which are useless in differential geometry. In my opinion, the definition of differentiable manifold structure most appropriate for differential geometry is using the latter definition and identifying the maximal atlas with the atlases that can represent it. That is, we should ignore the differences among them, but keep the distinction between them and the rest of atlases in the equivalent class. In differential geometry, we should accept set theory flexibly; in other words, we should transform it to a tool useful in differential geometry. Furthermore, for the atlas that can represent the
maximal atlas, we keep a minimum number of charts in it as long as they are good enough for practical use.

Remark 3. The maximal smooth atlas $\mathcal{A}$ constructed in the proof of Lee [66, p.13, Proposition 1.17(a)] does not seem to use the axiom of choice. However, if we consider the proof details more carefully, we will find that the construction of $\mathcal{A}$ actually uses the axiom of choice. $\mathcal{A} = \{ (U, \varphi) \text{ on } M :\text{for every chart } (W, \theta) \in \mathcal{A}, (U, \varphi) \text{ is smoothly compatible with } (W, \theta) \}$.

If $\{ \text{charts on } M \}$ is uncountable, then the candidates to be tested are uncountable. If $\mathcal{A}$ is uncountable, then each candidate must be tested uncountable times to see if it satisfies the required condition. Consequently, there are cases such that we cannot complete the construction of $\mathcal{A}$ without using the axiom of choice twice.

Example 6.117. (Differentiable manifolds vs. locally compact Hausdorff spaces [Spivak [95] vol.1, p. 44, Lemma 2]; Rudin [88] p.40, Lemma 2.12])

Spivak [95] vol.1, p. 44, Lemma 2] and Rudin [88] p.40, Lemma 2.12] are theorems of the same type. The setting of the former one is a Differentiable manifold, while the setting of the latter one is a locally compact Hausdorff space. In order to obtain $f$ for the former theorem, we need to construct only three functions $j, g, l$ of Spivak [95, vol.1, p. 43]. In contrast, in order to obtain $f$ for the latter theorem, we need to construct a sequence of functions; see Rudin [88, p.41, (5)].

Example 6.118. (The strong version of Sard’s theorem [Spivak [95, vol.1, p. 55, Theorem 8]; Sternberg [96, p.47, Theorem 3.1]])

The purpose of this example is to make the proof of Sternberg [96, p.47, Theorem 3.1] readable. There are several barrier gates need to break through:

1. The observation $\psi^r = \psi^r(u^1, \cdots, u^k)$ given in Sternberg [96, p.41, l.−18] is deeper than Spivak [95, vol.1, p. 59, Theorem 10 (1)].

2. The purpose of Sternberg [96, p.39, l.−3–p.40, l.4] is to show that $(\varphi \text{ is differentiable}) \Rightarrow (h \circ \varphi \circ g^{-1} \text{ is differentiable})$ [Sternberg [96, p.40, l.2]] is differentiable). The purpose of Sternberg [96, p.40, l.4–l.11] is to show that $(h \circ \varphi \circ g^{-1} \text{ is differentiable}) \Rightarrow (\varphi \text{ is differentiable}).$

3. Sternberg [96, p.46, Exercise 3.2].

4. By Rudin [88] p.185, Theorem 8.26(d); p.186, l.−5], $\psi(V \cap A)$ will have measure zero [Sternberg [96, p.46, l.−13]].


Proof. \[ \forall x \in A \exists U_x : x \in U_x. \]
\[ \Rightarrow \exists \text{ ball } B_x : x \in B_x \subset U_x. \]
\[ A \subset \bigcup B_x \]
\[ = \bigcup B_x \text{ [since } E^n \text{ has a countable basis].} \]


7. By Definition 3.2, we can reduce the theorem to the case that $M_2$ is a Euclidean space, $M_1$ is a subset of unit cube $C : \{ x \in E^n | 0 \leq x_i \leq 1 \}$ and $f$ is a $C^k$ map of some neighborhood of $C \rightarrow E^{n_2}$ [Sternberg [96] p.47, l.−11–l.−8]].
Proof. I. By Sternberg [96, p.46, (3.4)], a diffeomorphism preserves sets of measure zero.
II. Let \((U_i,x_i)\) be a countable basis of \(M_1\); let \((V_j,y_j)\) be a countable basis of \(M_2\).
Since the rank of \(f\) at \(p \in M_1\) equals the rank of \(y_j \circ f \circ x_i\) at \(x_i(p)\), it suffices to prove that 
\[\{x_i(p) | x_i(p)\text{ is a critical point of }y_j \circ f \circ x_i^{-1}\}\] has measure zero. Consequently, we may assume \(M_2 = \mathbb{R}^n\).
III. Let \(C_p\) be the coordinate cube with center at \(x_p(p)\) and side length \(\varepsilon_p\) such that \(C_p \subset x_p(U_p)\).
Let \(C'_p\) be the coordinate cube with center at \(x_p(p)\) and side length \(\varepsilon_p/2\).
\(M_1 = \bigcup_p x_p^{-1}(C'_p)\).
Since \(M_1\) has a countable basis, \(\exists\) a sequence \(\{i\}\) such that \(M_1 = \bigcup_i x_i^{-1}(C'_i)\). Hence we may suppose \(M_1\) is a subset of unit cube \(C\).

8. Sternberg [96, p.48, 1.5, (3.6)] follows from Struik [98, p.55, (1-2)].
9. \(x \in A \setminus A_0 \Rightarrow f'(x) = f'(2)(x) = \cdots = f'(q) = 0\) [Sternberg [96, p.49, l.−9–l.−8]].

Proof. There exist \(x_n^{(1)}\) such that \(f'(x_n^{(1)}) = 0\) and \(x_n^{(1)} \to x\). By the continuity of \(f'\), \(f'(x) = 0\).
\[\cdots\]
There exist \(x_n^{(q)}\) such that \(f'(q)(x_n^{(q)}) = 0\) and \(x_n^{(1)} \to x\). By the continuity of \(f'(q)\), \(f'(q)(x) = 0\).

10. By the uniform continuity of \(f'(q)\) on a compact set, (3.13) is a consequence of Taylor’s formula [Sternberg [96, p.49, l.−7–l.−6]].
11. “By (3.11)” given in Sternberg [96, p.50, l.112] should have been replaced with “By an argument similar to the one given in Sternberg [96, p.49, l.−9–l.−6]”.
12. \(\sqrt{n_1n_2b(\frac{\sqrt{n_1}}{p})(\frac{\sqrt{n_1}}{p}^q)\} [Sternberg [96, p.50, l.−11]] should have been replaced with \(\sqrt{n_2b(\frac{\sqrt{n_1}}{p})(\frac{\sqrt{n_1}}{p}^q)\}\).
13. \(K = \sqrt{n_1n_2(\frac{\sqrt{n_1}}{p})^{q_2}\omega_{n_2}\} [Sternberg [96, p.50, l.−8]] should have been replaced with \(K = (\frac{\sqrt{n_2}}{p})^{q_2}(\frac{\sqrt{n_1}}{p})^{q_2}\omega_{n_2}\).
14. \(q \geq (n_1 − r)/(n_2 − r), (0 \leq r < n_2)\) [Sternberg [96, p.52, l.1.19]].

Proof. By Sternberg [96, p.47, l.−13], we may assume \(n_1 \geq n_2\).
\(q \geq n_1 − n_2 + 1\) [Sternberg [96, p.47, (3.5)]. \(q(n_2 − r) \geq (n_1 − n_2 + 1)(n_2 − r)\)]
\[= (n_1 − n_2)(n_2 − r) + (n_2 − r)\]
\[\geq n_2 − r\].

15. “By Lemma 3.1” given in Sternberg [96, p.50, l.−2] should have been replaced with “By an argument similar to that of Lemma 3.1”.
16. In Sternberg [96, p.52, l.1.16–l.20], \(f = \varphi\). “Lemma 3.4” given in Sternberg [96, p.52, l.1.16] should have been replaced with “Lemma 3.5”.
17. (3.20) is just uniform continuity because \(f \circ \varphi_i\) is uniformly continuous on \(B_{\varepsilon_i}^n\) and (3.20) becomes \(|f(\varphi_i(x)) - f(\varphi_i(y))| = b_i(||x - y||)\) in this case.
18. Case \(n = 1\) [Sternberg [96, p.53, l.4–l.5]].

Proof. Let \(A_0\) be the set of all isolated points of \(A\).
\(x \in A \setminus A_0 \Rightarrow f(x) = f'(x) = f'(2)(x) = \cdots = f'(q) = 0\).
Take \(\varepsilon_i \to 0\), \(A_i = A \cap \bar{B}(0, \varepsilon_i)\) and \(\varphi_i\) be the identity map.
19. We may assume \( n > 1, k > 0 \) [Sternberg [96, p.53, 1.7]] because case \( k = 0 \) is proved in Sternberg [96, p.53, 1.1–1.4] and case \( n = 1 \) is proved in Sternberg [96, p.53, 1.4–1.5].

20. By Sternberg [96 p.53, (3.21)], (3.22) holds for \( g = \frac{\partial f}{\partial x_i} \) if \( f \in C^k \) and vanishes on \( A \) [Sternberg [96 p.53, 1.1–1.15]].

21. \( K_1 = n \max_{x \in B^n, 1 \leq \alpha \leq m, 1 \leq j \leq n} |\partial \varphi^j / \partial x^\alpha| \) [Sternberg [96 p.54, 1.1]] should have been replaced with \( K_1 = m \max_{x \in B^n, 1 \leq \alpha \leq m, 1 \leq j \leq n} |\partial \varphi^j / \partial x^\alpha| \).

22. “There is some function \( g \) vanishing on \( A \) with some \( \partial g / \partial x_i(p) \neq 0 \)” [Sternberg [96 p.54, 1.8–1.9]] should have been replaced with “There is some function \( g \) \( \in C^k \) vanishing on \( A \) with some \( \partial g / \partial x_i(p) \neq 0 \).”

23. By Widder [111 p.19, Theorem 7], \( \varphi \in C^k \) [Sternberg [96 p.54, 1.18]].

24. \( N \cap A \subset \varphi(B_{1}^{n-1}) \) [Sternberg [96 p.54, 1–16]].

Proof. If we write \( A \) and \( B \) in Spivak [94, p.41, Theorem 2-12] as \( A_C \) and \( B_C \) respectively, then \( A_C \) is the ball \( B_{1}^{n-1} \) with center at \( (p^1, \ldots, p^{a-1}) \) and \( B_C \) is an open set containing \( p^n \) such that \( \varphi^p(x) \in B_C \iff g(\varphi(x)) = 0 \).

Let \( N = A_C \times B_C \).

Then \( N \cap A \subset N \cap \{ q \in B_{1}^{n-1} | g(q) = 0 \} = \varphi(B_{1}^{n-1}) \).

25. By Spivak [94 p.35, Theorem 2-11], \( \varphi^{-1} \) is continuously differentiable [Sternberg [96 p.54, 1–16–1.15]].

26. \( \psi_r = \varphi \circ \psi_r \) satisfies (3.19) [Sternberg [96 p.54, 1–14–1.13]].

Proof. \( ||\varphi \circ \psi_r(x) - \varphi \circ \psi_r(y)|| \geq ||\psi_r(x) - \psi_r(y)|| \) [the former has one more component than the latter] \( \geq ||x - y|| \) [by the induction hypothesis], where \( x, y \in B_{1}^{n-1} \).

27. \( A \subset \cup_i A_i \) given in Sternberg [96 p.49, 1.16; p.52, 1–8] should have been corrected as \( A = \cup_i A_i \).

28. We have a decomposition of \( N \cap A \) of the type desired in Lemma 3.5 [Sternberg [96 p.54, 1–9–1.8]].

Proof. I. \( N \cap A \subset \cup_i \varphi(D_r) \).

\( D_r \subset \psi_r(B_{1}^{n-1}) \), where \( \psi_r : B_{1}^{n-1} \rightarrow E_{m-1} \). Then \( \varphi(D_r) \subset \varphi \circ \psi_r(B_{1}^{n-1}) \).

Let \( A_r = \varphi(D_r) \). Then \( A_r \subset \varphi_r(B_{1}^{n-1}) \).

By the induction hypothesis, \( \psi_r \) is a homeomorphism; by Spivak [95, vol.1, p. 3, Theorem 1], \( \varphi \) is a homeomorphism; see §3 Generalizations in [https://en.wikipedia.org/wiki/Invariance_of_domain]. Consequently, \( \varphi_r \) is a homeomorphism.

II. \( |f \circ \varphi_r(x)| = |(f \circ \varphi) \circ \psi_r(x)| \leq b_r(||x-y||)||x-y||^k \) [by the induction hypothesis].

29. Lemma 3.3 follows from Lemma 3.6 [Sternberg [96 p.54, 1–5–1.4]].

Proof. For every \( x \in A, x \) is a critical point of \( f \). Then \( \frac{\partial f}{\partial x_i}(x) = 0 \) [Otherwise, \( x \) would not be a critical point of \( f \)].

\[ |\frac{\partial}{\partial x_j} (\varphi_i(x))| < b_i(||x-y||)||x-y||^k \], where \( x, y \in B^m, \varphi_i(y) \in A_i \) [Sternberg [96 p.52, Lemma 3.6]].

\[ |f(\varphi_i(x)) - f(\varphi_i(y))| < K b_i(||x-y||)||x-y||^k \], where \( \varphi_i(x), \varphi_i(y) \in A_i \) [Sternberg [96 p.53, Lemma 3.7]].
Example 6.119. (The indirect solving method by studying the problem’s background first sheds more insight on why we solve the problem this way)

The claim "\(C_1\) is a closed set" given in Spivak [95, vol. 1, p.68, l.2–l.4] is incorrect because \(U_1\) is not closed. However, I do not want to correct Spivak’s mistake directly. I want to discuss first the general attitude toward proving a theorem in topology in a textbook of differential geometry. Dugundji [28, p.311, ⇒ diagram] provides a bird’s eye view of the related theorems in topology. From this diagram, we see \(\sigma\)-compact ⇒ paracompact. Spivak [95, vol. 1, p.67, Theorem 14] follows from Dugundji [28, p.152, Theorem 6.1; p.162, Definition 2.1]. We should avoid using the axiom of choice the best we can. For example, Spivak [95, vol. 1, p.6, Theorem 2] and (\(\sigma\)-compact ⇒ regular Lindel" of) [Dugundji [28, p.311, ⇒ diagram]] allow Spivak [95] to avoid using the axiom of choice. However, Dugundji [28] fails to follow this practice. Consequently, it is better to go back to correct Spivak’s mistake. The method comes from the proof of Dugundji [28, p.152, Theorem 6.1, (1) ⇒ (2)]:

It suffices to construct by induction on \(k\) a sequence of open sets \(V_k(k \in \mathbb{N})\) satisfying the following requirements:

For \(V_i(i \leq k)\),

(a). \(\overline{V}_i \subset U_i, V_i \neq \emptyset\) whenever \(U_i \neq \emptyset\).

(b). \(\{V_i|i \leq k\} \cup \{U_i|i > k\}\) is a covering of \(M\).

Suppose (a) and (b) are valid for \(i < k\). Now we want to define \(V_k\).

Let \(F = M \setminus (\bigcup_{i < k} V_i \cup \bigcup_{j > k} U_j) \subset U_k\). \(F\) is closed.

By Dugundji [28, p.144, Theorem 3.2, (1) ⇒ (2)], there exists an open set \(V_k\) such that \(F \subset V_k \subset \overline{V}_k \subset U_k\).

If \(F = \emptyset\), replace \(F\) with a point in \(U_k\).

Then (a) and (b) are valid for \(i \leq k\). Thus, the sequence \(V_i(i \in \mathbb{N})\) is defined by induction.

This indirect approach that I have adopted sheds more insight on why we solve the problem this way than the direct approach; this correction method is typical.

Example 6.120. (Natural viewpoints vs. unnatural viewpoints toward tangent bundles)

Mathematical development tends to become simple and natural. We do not care how many contents a textbook provides, how difficult it is to read these contents, or how odd the viewpoint that the author adopts is compared with the standard one, but we do care about if the author adopts the natural viewpoints to discuss the topic because the natural viewpoints make it easy to see the big picture.

I. On a differentiable manifold \(M\), the following four statements are equivalent:

(a). \(v_p[f] = \frac{d}{dt} f(p + tv)|_{t=0}\) [O’neill [74, p.11, Definition 3.1]].

(b). \(\alpha'(t)[f] = \frac{d(f(\alpha(t))}{dt}(t)\) [O’neill [74, p.19, Lemma 4.6]].

(c). \(v_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(p)\) [O’neill [74, p.12, Lemma 3.2]].

(d). \(v_p[f]\) satisfies the properties given in O’neill [74, p.12, Theorem 3.3].

Proof. Since our goal is to highlight the main ideas rather than the details, in some formulations and proofs I consider only the case \(M = \mathbb{R}^n\). The formulations and proofs for the general case are similar.

O’neill [74] p.11, Definition 3.1; p.19, Lemma 4.6; p.12, Lemma 3.2 & Theorem 3.3] show (a)⇒(b)⇒(c)⇒(d).

Spivak [95, vol. 1, p.107, Theorem 3] shows (d)⇒(c).

Remark 1. All the arrows in the above proof are natural flows except the last one.

Remark 2. We construct \(TM\) [Spivak [95, vol. 1, p.103, l.1–7]] by duplicating the structure of the tangent
space of $E^3$ at $p$ [O'neill [74, p.7, Defintion 2.2]] to a differentiable manifold. We also generalize the derivative map $F_*: T_p(E^3) \to T_{f(p)}(E^3)$ [O’neill [74, p.36, l.5–l.10]] to $F_*: TM_p \to TN_{f(p)}$ [Spivak [95], vol. 1, p.104, l.9; (c)]]. Since the duplication or generalization is through coordinate charts, we must ensure that the results through various charts are compatible. O’neill [74, p.161, FIG. 4.33] provides a natural and the best explanation for $F_*$ and O’neill [74, p.146, Definition 3.5] provides the most suitable version of tangent vectors to describe this figure [O’neill [74, p.160, Definition 5.3]]. $T'M$ [Spivak [95] vol. 1, p.105, l.−2] is essentially the same as O’neill [74, p.146, Definition 3.5]. From the viewpoint of directional derivative, these unnatural and complicated explanations of $f_*$ are not as good as the natural explanation given in O’neill [74] p.161, FIG. 4.33].

Example 6.121. (The physical meaning of ODEs from the global view vs. that from a local view)
I. Physical meanings may inject new blood and new life into an abstract theorem in ODEs. They give its argument flow a guiding direction and concrete meanings. The physical meanings of Spivak [95] vol. 1, p.203, Theorem 5] are more clear and explicit from the global view; if we consider a local view alone, all we can see is odds and ends rather than the big picture. This is because a local view preserves only a small part of the global features. Without this big picture in mind, the discussion of Spivak [95] vol. 1, p.203, Theorem 5] would become merely a display of a mess of meaningless formulas. In fact, a local view would make features loom as if they are both hidden and present, seem as if there are both something and nothing. Thus, this would make us difficult to express them clearly and logically. If one tries, it might turn out to be a wasted effort. The global view: phase spaces [Arnold [4, p.15, l.−20–l.7]], one-parameter transformation groups [Arnold [4, p.60, l.11–l.24; p.60, l.−2–p.61, l.−13]], one-parameter diffeomorphism groups [Arnold [4, p.61, l.−11–l.−4; p.62, l.11–l.1]], the one-to-one correspondence between one-parameter diffeomorphism groups and ODEs built through phase velocity vector fields [Arnold [4, p.63, l.5–l.11; p.63, l.−5–p.64, l.12; p.64, l.−14–l.13]], the action of diffeomorphisms on vector fields [Arnold [4, p.70, l.5–p.71, l.3; p.71, l.7–l.−1]], change of variables in an equation with a diffeomorphism [Arnold [4, p.72, l.5–p.73, l.10]], the action of a diffeomorphism on a direction field [Arnold [4, p.73, l.18–p.75, l.4]], the action of a diffeomorphism on a phase flow [Arnold [4, p.75, l.5–p.76, l.12]], symmetry groups [Arnold [4, p.76, l.13–p.77, l.5]]. A local view: Arnold [4] p.78, l.9–l.9; this coordinate choice can be generalized to a manifold.

Remark 1. Spivak [95] vol. 1, chap. 3] gives three definitions of tangent vectors. The first one is given in Spivak [95] vol. 1, p.103, l.−7]. It has the advantage in coordinate representations. See Spivak [95] vol. 1, p.104, (c)]. The second one is given in Spivak [95] vol. 1, p.105, l.−8–p.106, l.6]. Its advantage lies in the fact that it is defined in a natural way. See O’neill [74, p.60, Definition 5.3; p.161, FIG. 4.33]. The third one is given in Spivak [95] vol. 1, p.106, l.7–p.109, l.−5]. Its advantages: Making it easy to prove $Xf \in C^\infty$ [Spivak [95] vol. 1, p.113, l.11–l.17)] or to calculate the corresponding result when the coordinate system is changed [Arnold [4, p.71, l.7–l.18]]. Listing various versions of a definition without discussing their individual advantages, then all the effort of introducing them is simply wasted.

Remark 2. Pontryagin [81] p.222, Figure 48] presents only the shape of a limit cycle, but fails to provide an effective method to construct it. In contrast, Arnold [4, p.72, l.10–p.73, l.10] gives an effective method of constructing a limit cycle step by step.

149

II. Rectification: Suppose the solutions of an ODE are known. We try to use diffeomorphisms to map the orbits of phase flow or integral curves of the direction field into curves of simple shapes. Examples are Arnold [4] p.78, Fig. 62; p.80, Fig. 64; p.89, Fig. 69.

Separation of variables: Suppose the solutions of an ODE are unknown. By a proper choice of new variables, we can use the method of separation of variables to solve the ODE. For example, for solving a homogeneous ODE with separation of variables, read §Homogeneous first-order differential equations of https://en.m.wikipedia.org/wiki/Homogeneous_differential_equation.

Thus the above two concepts are totally different. Except for Arnold [4, chap. 1, §6.6], Arnold [4, chap. 1, §6] essentially discusses the rectification of integral curves for homogeneous or quasi-homogeneous ODEs. Unfortunately, Arnold somehow mistakes rectification for separation of variables; see Arnold [4, p.76, l.14–l.17]. It is important that we should not consider Arnold [4, p.79, l.17–l.5, Theorem] [resp. Arnold [4, p.83, l.4–l.6, Theorem]] the method of separation for homogeneous [resp. quasi-homogeneous] ODEs because we should not use a theorem itself to prove the same theorem. In other words, we should not consider the existence of solution of an ODE proved by using change of variables to separate the ODE’s variables under the assumption that the solutions of an ODE are known. Here I pinpoint this assumption for Arnold [4, p.77, l.17–p.78, l.2, Theorem; p.79, l.6–l.5, Theorem; p.83, l.4–l.6]: Arnold [4, p.78, l.17–l.20; p.79, l.14–l.3; p.83, l.7–l.8].

III. The climax of Arnold [4, chap. 1, §6.5] is Arnold [4, p.85, Problem 6].

1. Proof of Arnold [4, p.84, Problem 1]: \(e^{rt}e^{st} = e^{(r+s)t}\).
2. Arnold [4, p.84, Problem 2].

Proof. \(v_{\text{new}} = \frac{d(e^{sx}e^{sy})}{dt} = e^{ms}(P(x), Q(y)) = e^{ms}(x, y) = e^{ms}v_{\text{old}}\).

\[T_{\text{new}} = \frac{1}{v_{\text{new}}} = \frac{e^{ls}v_{\text{old}}}{e^{ms}v_{\text{old}}} = e^{(1-m)s}T_{\text{old}}.\]

3. Arnold [4, p.85, Problem 3].

Proof. Along the same angle, the angular velocity at the phase point on \(g^s\gamma\) is \(e^{s\gamma}\) times of that at the corresponding phase point on \(\gamma\).

4. Arnold [4, p.85, Problem 5].

Proof. \(\alpha - 2\beta = ma\) [Arnold [4, p.80, Definition; p.84, Theorem]].

\(\frac{dy}{dx}_{\text{new}} = e^{(m-1)s} \frac{dy}{dx}_{\text{old}}\) [Using polar coordinates and Arnold [4, p.84, Problem 4]].

\(x_{\text{new}} = e^{2s}x_{\text{old}} (\lambda = e^{2s}) \Rightarrow T_{\text{new}} = e^{(1-m)s}T_{\text{old}}.\)

IV. Clarifications.

1. \(\sum x_i \frac{\partial f}{\partial x_i} = rf(x)\) [Arnold [4, p.81, l.5–l.8]]

Proof. Let \(g(\alpha) = f(e^{\alpha t}x)\).

\(g'(\alpha) = \sum e^{\alpha t} \frac{\partial f}{\partial x_i} (e^{\alpha t}x) = rf(e^{\alpha t}x)t = rg(\alpha)t\).

\(\frac{g'(\alpha)}{g(\alpha)} = rt\)

\(\Rightarrow g(\alpha) = g(0)e^{rt}.\)
2. The vector field \( \sum \alpha_i x_i \frac{\partial}{\partial x_i} \) is the phase velocity field of a group of quasi-homogeneous dilations.

**Proof.** \( \mathbf{v}(x_1, x_2) = \frac{g'(x_1, x_2)}{dt} |_{t=0} \) [Arnold [4] p.63, Definition]

\[
= \left( e^{a_1 t} x_1, e^{a_2 t} x_2 \right) |_{t=0} = (\alpha_1 e^{a_1 t}, \alpha_2 e^{a_2 t}) |_{t=0} = \sum \alpha_i x_i \frac{\partial}{\partial x_i} \text{[Spivak [95] vol. 1, p.108, l.4].}
\]

\( \square \)

**Example 6.122.** (A textbook author should not omit a proof simply because it takes a lot of trouble to write it down clearly)

A textbook author should not omit a proof simply because it takes a lot of trouble to write it down clearly. The author should provide at least the key idea of the proof. What readers need is methods rather than results. The omission of methods only leaves readers groping in the dark. Very frequently, a proof looks easy, but when one writes it out step-by-step, it may be not. There are also times when one finds problems that one may not foresee at first. Thus, the omission of a proof can easily hide errors. Here are two examples.

Example 1. (If one may not foresee at first. Thus, the omission of a proof can easily hide errors. Here are two examples.

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i (x_i)^n = \left( \sum_{i=1}^{n} \alpha_i x_i \right)^n \text{[Spivak [95] vol. 1, p.213, l.1].}
\]

Similarly, \( \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i (x_i)^n = \left( \sum_{i=1}^{n} \alpha_i x_i \right)^n \text{[Rudin [86] p.27, Theorem 1.34 or p.246, Exercise 16].}

**Proof.** \( g(t) = \int_{0}^{t} f'(s) ds \) [Spivak [95] vol. 1, p.107, l.1].

\[
g'(t) = \int_{0}^{t} f''(s) ds \text{[Rudin [88] p.27, Theorem 1.34 or p.246, Exercise 16].}
\]

Similarly, \( g''(t) = \int_{0}^{t} f'''(s) ds. \) Then \( (g') \) is differentiable in \( (-\varepsilon, \varepsilon) \)

\[
\Rightarrow g' \in C(-\varepsilon, \varepsilon) \text{[i.e. } g \in C^1(-\varepsilon, \varepsilon)].
\]

**Example 2.** (Computing \( \lim_{n \to \infty} f_h(x_h) \))

\[
\lim_{n \to \infty} (Yg_h)(\phi_{-h}(p)) = (Yg_0)(p) \text{[Spivak [95] vol. 1, p.214, l.1–l.5].}
\]

**Proof.** I. By Spivak [95] vol. 1, p.213, l.1–3 [“there is a function \( f' \) should have been replaced with “there is a function \( g' \)], \( g \in C^\infty. \) Then \( \frac{\partial g}{\partial x} \in C^\infty. \) Define

\[
Yg : (-\varepsilon, \varepsilon) \times U \to \mathbb{R}
\]

\[
(h, q) \mapsto Yg_h(q)
\]

, where \( U \) is a compact neighborhood of \( p \) and \( g_h(q) = g(h, q). \)

By Spivak [95] vol. 1, p.107, Theorem 1], \( Yg \in C^\infty((-\varepsilon, \varepsilon) \times U). \)

II. By Rudin [86] p.135, Theorem 7.11], in order to prove

\[
\lim_{n \to \infty} \lim_{h \to 0} (Yg_h)(\phi_{-h}(p)) = \lim_{h \to 0} \lim_{n \to \infty} (Yg_h)(\phi_{-h}(p)) \{\{g_n\} \text{ can be any sequence of } \{g_h\} \text{ satisfying}
\]

\( h_n \to 0, \) it suffices to prove that \( Yg_h \) converges uniformly on \( [-\varepsilon/2, \varepsilon/2] \times U. \)

This requirement follows from I because \( Yg \) is uniformly continuous on \( [-\varepsilon/2, \varepsilon/2] \times U. \)

\( \square \)

**Example 6.123.** (The integrability theorem envolves by step-by-step adding geometric meanings; as the level gets more advanced, its geometric meanings gets more generalized)

I. (a). In calculus, the integrability theorem is given as follows:

Spivak [95] vol. 1, p.250, l.1–3, (**) holds iff \( \exists ! \alpha \) satisfies Spivak [95] vol. 1, p.251, l.1–l.3, (**)].

(b). In PDE, the integrability theorem is given as follows:

Spivak [95] vol. 1, p.254, (**)] holds iff \( \exists ! \alpha \) satisfies Spivak [95] vol. 1, p.254, l.6–l.8, (**)].
The evolution is guided by the example of distribution given in Spivak [95 vol. 1, p.252, l.6, $\Delta_p$]: Spivak [95 vol. 1, p.253, (**)] holds iff there is an integral manifold of $\Delta$ through every point $p \in M$ [Spivak [95 vol. 1, p.253, l.7–l.13]]. Thus, we introduce the first concept of integral manifold of distribution [Spivak [95 vol. 1, p.246, l.6–p.247, l.2]]

By Spivak [95 vol. 1, p.260, l.7–p.261, l.3], $[X, Y] \in \Delta \iff [X, Y] = 0$, where $[X, Y] = 0$ iff Spivak [95 vol. 1, p.253, (**)] holds. Thus, we introduce the second concept of integrable distribution.

II. Spivak [95 vol. 1, p.254, Theorem 1] $\iff$ Spivak [95 vol. 1, p.262, Theorem 5] [Spivak [95 vol. 1, p.262, l.1–l.2]]

Proof. $\Rightarrow$: Let $N = \{q \in U : x^{k+1}(q) = a^{k+1}, \ldots, x^n(q) = a^n\}$. Then

$N = \{q \in U : \frac{\partial a(x^{l}, \ldots, x^n)}{\partial x^1} = 1, \ldots, \frac{\partial a(x^{l}, \ldots, x^n)}{\partial x^k} = 1; \alpha(0) = (0, \ldots, 0, a^{k+1}, \ldots, a^n)\}$ [Spivak [95 vol. 1, p.254, Theorem 1]; both this case and Spivak [95 vol. 1, p.254, Theorem 1] try to solve PDEs]

Fix $p \in M$. Then

$i_i(N_p) = \mathbb{R}^k$ [generated by $\frac{\partial}{\partial x^i}(i = 1, \ldots, k)$; O’neill [74, p.6, Definition 2.1; p.146, Definition 3.5] or Spivak [95 vol. 1, p.106, l.3–l.5] $\iff$ Spivak [95 vol. 1, p.256, l.10; consider dimension $k$]]. Consequently, $N$ is an integral manifold of $\Delta$, as stated in Spivak [95 vol. 1, p.263, l.1–l.10].

$\Leftarrow$: Spivak [95 vol. 1, p.276, Problem 6]: The conditions given in Spivak [95 vol. 1, p.254, (**)] are necessary because $\frac{\partial^2 a}{\partial x^i \partial x^j} = \frac{\partial^2 a}{\partial x^j \partial x^i}$. Consequently, it suffices to prove that the conditions are sufficient for the existence of solutions of Spivak [95 vol. 1, p.254, (*)].


Remark. In p.10, 1.14–1.6 of https://syafiqjohar.files.wordpress.com/2018/12/frobenius-1.pdf, if we define $\nu = (\partial_u, \partial_u, -1)$, then $X_1 = f_{11}\partial_1 + f_{12}\partial_1 + g_1\partial_u$, $X_2 = f_{21}\partial_1 + f_{22}\partial_1 + g_2\partial_u$. This argument not only explains the origin of $\Delta_p$ given in Spivak [95 vol. 1, p.252, l.6], but also shows that $\Delta$ is a distribution. Similarly, $\Delta_p$ given in Spivak [95 vol. 1, p.270, l.8] originates from Hicks [51, p.127, l.6].

Note that the expression of $Y_r$ given in Hicks [51, p.127, l.6] is simpler than the one given in Spivak [95 vol. 1, p.252, l.6]. Unless for the convinence of calculation, as in the case of Hicks [51, p.127, l.10], we need not use the complicated partial-derivative notation.

III. In terms of logic, Spivak [95 vol. 1, p.254, Theorem 1] and Spivak [95 vol. 1, p.262, Theorem 5] are equivalent. However, in terms of mathematics, the latter theorem is enriched by the mathematical structures of differentiable manifold, coordinate systems, integrable distributions, etc.

IV. Both Spivak [95 vol. 1, p.204, l.9–p.263, l.1–l.1] and Hicks [51, 9.1] discuss the Frobenius theorem. However, the essence of this topic contains only the following three theorems:


Theorem B: The $\Leftarrow$ part of Theorem 3.8 of https://syafiqjohar.files.wordpress.com/2018/12/frobenius-1.pdf.

Theorem C: Hicks [51 pp.126–127, Theorem].

The proof of Hicks [51 p.124, Theorem] is not as natural as the proof of Theorem B. The statement of Theorem A is closely connected with the proof of Theorem B, while the statement of Spivak [95 vol. 1, p.219, Theorem 14] is not. The set of PDEs considered in §4 in https://syafiqjohar.files.wordpress.com/2018/12/frobenius-1.pdf is a special case of the one given in Hicks [51 p.127, l.1]. For the proof of “Theorem C $\Rightarrow$ Theorem B", Spivak [95 vol. 1, p.262,l.1–p.263, l.1–l.1] says a lot,
but fails to grasp its essence; ditto with the proof given in Hicks [51, p.127, l.1–7–p.128, l.6]. In contrast, the following proof based on Theorem A and Theorem B is more clear and concise. The proof of Theorem B constructs the desired integral manifold $F(\Omega)$ using Theorem A which is based on the PDEs
\[
\frac{\partial \alpha(i)}{\partial t_1} = 1, \cdots, \frac{\partial \alpha(i)}{\partial t_1} = 1; \alpha(0) = 0.
\]

**Example 6.124.** (Characteristic property of a quotient structure vs. construction methods of the quotient structure)

I. Characteristic property of a quotient structure:

A. Lee [66, p.605, Theorem A.27 (a) & (b)] belongs to the general type given in Bourbaki [15, p.280, l.15–l.26]: Lee [66, p.605, Theorem A.27 (a)] corresponds to Bourbaki [15, p.280, (FI)] and Lee [66, p.605, Theorem A.27 (b)] corresponds to Bourbaki [15, p.280, CST18]. “each $g_i$ is a morphism of $A_i$ into $E$” [Bourbaki [15 p.280, l.1–6–l.5]] follows from Bourbaki [15, p.273, l.1–7–l.5].

B. For a particular mathematical structure like topology, we may have a more effective criterion to characterize quotient topology: Pervin [80, p.153, l.2–l.5].

C. In order to give Bourbaki [15, p.280, (FI)] a natural look, we may have the following view:

The statement given in Bourbaki [15, p.280, l.1–18] classifies, organizes, and summarizes the information given in Bourbaki [15, p.280, l.1–16].

Remark 1. Strictly speaking, Lee [66, p.309, Proposition 12.7] is a generalization [Lee [66, p.605, l.1–5]] rather than an example of quotient structure because $A$ is multilinear rather than linear. However, the underlying idea of tensor product space and quotient topology is the same, so their theory developments are similar. The characteristic property of tensor product space does not directly prescribe any construction method of tensor product space, but the resulting tensor product space by any construction method cannot violate the characteristic property.

II. Properties of quotient morphism:

(1). [Lee [66, p.605, Theorem A.30]] Given

\[
\begin{array}{ccc}
X & \xrightarrow{F} & B \\
\pi \downarrow & & \downarrow \\
Y
\end{array}
\]

; $\pi(p) = \pi(q) \Rightarrow F(p) = F(q)$. Then there exists a morphism $F : Y \to B$ such that

\[
\begin{array}{ccc}
X & \xrightarrow{F} & B \\
\pi \downarrow & & \downarrow \\
Y
\end{array}
\]

Remark 2. Lee [66, p.311, l.1–18–l.12] follows from Lee [66, p.311, Proposition 12.10].

(2). [Lee [66, p.606, Theorem A.31]] Given

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_1} & Y_1 \\
\pi_2 \downarrow & & \downarrow \\
Y_2
\end{array}
\]

; $\pi_1(p) = \pi_1(q) \Leftrightarrow \pi_2(p) = \pi_2(q)$. Then there exists an isomorphism $\varphi : Y_1 \to Y_2$ such that

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y_2 \\
\pi_1 \downarrow & & \downarrow \\
Y_1 & \xleftarrow{} & Y_2
\end{array}
\]

Remark 3. (2) follows from (1); see the proof of Lee [66, p.606, Theorem A.31].

III. Both set theory and category theory discuss mathematical structures [Bourbaki [15 chap. IV]; Lee [66, p.73, l.1–8–p.75, l.1–18]]. Lee [66, p.74, l.1–12–l.1–3] may establish the relationships between two objects or two morphisms belonging to different categories, so the language of category theory is wider and more
appropriate for discussing mathematical structures than set theory.

**Example 6.125.** (Recovery of skills in definition design)

I. In an axiomatic system, we give axioms and definitions first, and then derive theorems from them. Thus, in an axiomatic approach to developing a theory, we must have the foresight of making it consistent with the existing theory when introducing a new definition. The belief of its truth for readers is supposed to form in the future. However, a definition is usually given without any explanation. Its legality relies on the rationale that you will not get a contradiction as you proceed. In order to put it on a more solid foundation, we should not blindly accept it. How can we predict its truth? How can we find clues for its justification? In other words, we should ask how the definition is designed. That is, we should recover the skills of definition design.

II. The definition of exterior derivative given in Spivak [95, vol. 1, p.286, l.1–l.6] is natural and intuitive because the $p$-form is expressed in the standard basis. All we need to do is take the differential of each coefficients. In contrast, the definition of the exterior derivative of 1-form given in O’neill [74, p.154, Definition 4.4] requires a justification because the form fails to be expressed in the standard basis [Spivak [95, vol. 1, p.279, Theorem 3]] so that the above underlying universal principle fails to be revealed.

III. The justification of O’neill [74, p.154, Definition 4.4].

Let $\phi = f_1dx^1 + f_2dx^2$ [Spivak [95, vol. 1, p.279, Theorem 3]].

$$d\phi = \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2}\right)dx^1 \wedge dx^2$$ [Spivak [95] vol. 1, p.286, l.5].

$$d\phi\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right) = \frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2}$$

$$= \frac{\partial}{\partial x^1}\left(\phi\left(\frac{\partial}{\partial x^2}\right)\right) - \frac{\partial}{\partial x^2}\left(\phi\left(\frac{\partial}{\partial x^1}\right)\right).$$

**Example 6.126.** (Telling the nuances between an algebraic dual spaces and a Banach dual space to clarify confusion)

Being confused means that there is something that one needs to learn. If one understands the statement and the proof of a theorem, one will say that one understands the theorem. This is not quite true. To test one’s understanding, a second theorem with similar hypotheses and opposite conclusion should be brought in and let one tell the nuances between the two to explain why the two theorem do not contradict each other. If one does not know what to do, this reveals that one’s understanding is shallow. In other words, consistency and thorough understanding are important. p.1, l.13–l.22 in [https://kconrad.math.uconn.edu/blurbs/linmultialg/dualspaceinfinite.pdf](https://kconrad.math.uconn.edu/blurbs/linmultialg/dualspaceinfinite.pdf) provides such a case: it compares an infinite-dimensional vector space $V$ and $l^2$ and explains why $\dim V < \dim V^{**}$ and $l^2 \cong l^{2**}$ do not contradict each other.

Since the algebraic dual is not as natural as the Banach dual space, we just need read [https://kconrad.math.uconn.edu/blurbs/linmultialg/dualspaceinfinite.pdf](https://kconrad.math.uconn.edu/blurbs/linmultialg/dualspaceinfinite.pdf) once. The prerequisites are Dugundji [28, p.52, l.4; p.47, Corollary 7.7; p.48, the proof of Theorem 7.8(2)]. It is inappropriate to express the Theorem in [https://kconrad.math.uconn.edu/blurbs/linmultialg/dualspaceinfinite.pdf](https://kconrad.math.uconn.edu/blurbs/linmultialg/dualspaceinfinite.pdf) as an exercise [Lee [66, p.620, Exercise B.5]] because the information provided is inadequate. All we need is a reference for finding the detailed solution of this exercise.

**Example 6.127.** (Different stances may make discussion get stuck and leave questions unanswered)

I. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is $C^\infty$. Then the notation $D_{f, v}$ may have the following two meanings:

1. The first meaning: (the matrix $(D_{j,f})(x)$ of the differential $D_{f,x}$) [Rudin [86, p.191, l.17–l.18]] × (the column vector $v$).

2. The second meaning is given in O’neill [74, p.23, Definition 5.2].

II. The first meaning and the second meaning are equivalent.
**Proof.** We write $f'$ given in Rudin [86, p.188, Definition 9.10] as $Df$.

By Rudin [86, p.191, l.17–l.18], $Df_{x}v = (D_{1}f, D_{2}f, D_{3}f) \times (v_{1}, v_{2}, v_{3}).$

By O’neill [74, p.12, Lemma 3.2], the quantity on the right-hand side of the above equality is just the directional derivative $v_{p}[f]$.

Remark 1. The above argument can be used to prove the cases when $\mathbb{R}^{3}$ is replaced with $\mathbb{R}^{n}$ or with a smooth manifold $M^{n}$.

Remark 2. The proof given in II gives the underlying reason why O’neill [74, p.23, Definition 5.2] is defined that way.

III. (Different stances may make discussion get stuck and leave questions unanswered)

The scenario of https://math.stackexchange.com/questions/1120430/derivative-of-bilinear-forms is as follows: Let Q be the one who proposes the question and A be the one who answers the question.

Q: The notation $D_{f}(x,y)(a,b)$ means the first meaning to me. Since you interpret it as the second meaning, you fail to answer my question.

A: According to O’neill [74, p.23, Definition 5.2], $D_{f}(x,y)(a,b)$ means the second meaning. Consequently, I completely answer your question.

Q’s view: A’s answer is unsatisfactory because he fails to prove that $f$ is differentiable. A should have proved the differentiability of $f$ to validate his original argument in https://math.stackexchange.com/questions/1120430/derivative-of-bilinear-forms.

A’s or someone else’s view: It is O’neill [74] that should be blamed because it fails to prove the equivalence of the notation’s two possible meanings.

The discussion has gotten stuck and the questions have been left unanswered ever since.

IV. Lee [66, p.643, Exercise C.2(g)] is better proposed so that A won’t have any excuse to avoid proving the differentiability of the bilinear map $B$. In order to prove the differentiability of $B$, it suffices to consider the following case: $V = \mathbb{R}^{m}, W = \mathbb{R}^{n}, X = \mathbb{R}$.

**Proof.** 1. Let $A_{ij} = B(e_{i}, e_{j})$. Then $B(v, w) = (v_{1}, \cdots, v_{m})(A_{ij}) \begin{pmatrix} w_{1} \\ \vdots \\ w_{n} \end{pmatrix}$.

2. $B(v + \Delta v, w + \Delta w) - B(v, w) = [B(v + \Delta v, w + \Delta w) - B(v, w + \Delta w)] + [B(v, w + \Delta w) - B(v, w)]$

By 1, the second term can be ignored as $\Delta v, \Delta w \to 0$ if we compare it with the other two terms.

$B(\Delta v, w) = \begin{pmatrix} \Delta v_{1} \\ \vdots \\ \Delta v_{m} \end{pmatrix} (A_{ij}) \begin{pmatrix} w_{1} \\ \vdots \\ w_{n} \end{pmatrix}$ is the linear transformation $B(\cdot, w)$ on $\mathbb{R}^{m}$.

$B(v, \Delta w) = \begin{pmatrix} v_{1} \\ \vdots \\ v_{m} \end{pmatrix} (A_{ij}) \begin{pmatrix} \Delta w_{1} \\ \vdots \\ \Delta w_{n} \end{pmatrix}$ is the linear transformation $B(v, \cdot)$ on $\mathbb{R}^{n}$.

**Example 6.128.** (One may increase reading efficiency for a tool book by 88 times if one has a goal in mind)

A wrench is useless until one uses it to repair a pipe leak. The theorems in a tool book do not have

155
meanings; the meaning of a theorem appears only when one uses it. In my opinion, a tool book, like a tool room, should provide a tool’s location and properties (usage). It should not contain any exercise. This is because most methods in a tool book are stereotype and the original idea for these methods can only be found in a broader and more inspiring area. Thus, a tool book should provide at least the exact location of solutions if it contains any exercise. Someone may say exercises help one’s thinking. Well, there are a lot of better things to do than solving exercises in a tool book.

When I was a university student, it took me six months to read Bourbaki [15, part 1, chap. 1, §1–§2]. Then I decided to read other easier topology textbooks like Pervin [80] and Dugundji [28] instead. Now I need to solve Lee [66, p.611, Exercise A.54]. I have found that the solution is given in Bourbaki [15, part 1, chap. 1, §10, no. 1; no. 2]. If I read the entire content of Bourbaki [15, part 1, chap. 1, §3, no.1–§10, no.2] aimlessly and indiscriminately, based on my past reading speed, it may take me at least 22 months to complete this task. It may not leave any impression in a little while. However, this time I just need to solve Lee [66, p.611, Exercise A.54], so I may avoid reading any theorem unrelated to this purpose. If I need to use a theorem, I can read only the small section containing that theorem. In this way, I solve the exercise in a week. In fact, I have read Bourbaki [15, part 1, chap. 1, p.37, l.9–p.37, l.17; p.43, l.6–p.45, l.14; p.47, l.9–p.48, l.12; p.50, l.12–p.52, l.11; p.56, l.4–l.12; p.57, l.8–p.65, l.15; p.68, l.1–p.72, l.9; p.72, l.15–p.73, l.10; p.74, l.1–l.18; p.75, l.1–p.77, l.18; p.83, l.7–p.85, l.3; p.97, l.11–p.100, l.8; p.101, l.6–p.103, l.8; p.104, l.9–l.14]. 1 week: 22 months = 1 : 88. Thus, if one has a goal in mind, one may increase reading efficiency by 88 times.

References


157


[39] Gibbon, J. D.: [http://www2.imperial.ac.uk/~jdg/AE2MAPDE.PDF](http://www2.imperial.ac.uk/~jdg/AE2MAPDE.PDF)


Mr. Li-Chung Wang is the author of the following website about the philosophy of mechanics:
http://www.lcwangpress.com/physics/main.html
Address: 7th Floor, #21 Lane 267, Xi-zhou Street, Chungli, Taiwan, ROC.
E-mail: lcwangpress@yahoo.com