Ordinary Differential Equations

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Abstract

When studying ordinary differential equations, we should aim at a higher level, partial differential equations, so that our approach to the study is directed toward that goal. We may discover that there is one principle which runs through the entire theory of differential equations if we have the right attitude toward solving them. In order to solve differential equations, we may want to classify differential equations by forms at first. In other words, we would like to create a table of solutions for differential equations. The more entries the table contains, the better it seems to be. In fact, this is not always true. For Bessel's equation alone, it has countless variations. No advantage is to be gained by studying these variations rather than Bessel's equation. Consequently, rather than by superficial forms, we should classify differential equations by essence: ordinary points, regular singularities, and irregular singularities. The rest of all the classifications and theories of differential equations can be considered supplements or generalizations of this view. We will discuss specific methods of solving ODEs, linear ODEs vs. linear systems (homogeneous case, Wronskians, reduction, adjoints, Green's second identity, nonhomogeneous case), the existence and uniqueness of solutions of ODEs, *maximal intervals of existence*, maximal or minimal solutions, sequences of ODEs, asymptotic behavior of solutions of an ODE, stability from the viewpoint of effectiveness, singularities, indicial equations, the Fuchsian type, self-adjoint eigenvalue problems, comparison theorems, Sturm's oscillation theorems, Green's functions, characteristic numbers for the Sturmian system, characteristic numbers for periodic boundary conditions where the coefficients of the system's equation are periodic, stable regions, self-adjoint boundary-value problems for second-order singular equations, limit-point type, limit-circle type, the expansion and completeness theorem, Riccati's equation, duality of adjoint systems, solving high order differential equations with Lommel's formula, finding the cases that Riccati's equation is integrable in finite terms, studying Sturm–Liouville Problems with the effective Ritz method, and miscellaneous notes.
Suppose we want to prove the equality of two functions with parameters. We first find the two differential equations that they satisfy and then manage to transform one of these differential equations to the other. This is the most powerful and effective method in analysis. The method of mathematical induction and all the methods in complex analysis are not competent enough for this task.
Methodical solutions: First, consider a differential equation of a special type. If its integral solution is based on guess, luck, and trial-and-error, we do not know from where the integrand comes, and the only way to justify the solution is by substitution, then this underdeveloped solution cannot be consider a methodical solution. Suppose the same equation also belongs to the wider class of equations of Laplacian type. In contrast, its integral solution can be built by a systematic method. In fact, the integrand and the path of integration can be specified by the Laplace transform. Consequently, the latter solution is more methodical than the former one.
"Applications of analytic continuation to the Weber–Schafheitlin integral (the right timing for a statement's appearance): Suppose we choose the weakest possible conditions required in an argument to be our theorem's hypothesis. If the argument has used the method of analytic continuation no more than once, then no confusion will occur. However, what should we do if the argument has used the method of

analytic continuation more than once?

Proposing a new condition without collecting enough evidence in advance has a problem with the timing for its appearance. Therefore, whenever we use the method of analytic contiuation, we should check and record if the change of the condition is needed so that we may easily clarify the relationship between cause and effect in the proof structure."

"Integration on a Riemann surface with branch points: If we reduce a contour integral on a Riemann surface to an integral along a line segment, the value of the latter integral may depend on which sheet the line segment is in, while the former integral is an invariant quantity. When we reduce a contour integral on a Riemann surface to an integral along a line segment, we often have to degeneate a part of the contour to a point. In order to make the argument of points along the contour continuous and simplify the calculation of these arguments, we should restore the degeneated point to its corresponding nondegeneate part."

"Contour integrals for Bessel functions"

"Carlini's solution of the Bessel equation for functions of large order"

"Evaluation of $\int_0^\infty \sin t \cdot t^{n-1} dt$ by using the Cauchy–Kowalevski theorem"

"The transient solution and the steady state solution of Wangsness [39, p.453, (27-11)]. We would like to see what we would miss if we were to study the complementary (transient) solution and the particular (steady state) solution from the viewpoint of ODEs alone."

Keywords. Specific effective methods, homogeneous, Euler linear equation, differentiation under the integral sign, Liouville's theorem, Wronskians, variation of constants, method of successive approximation, reduction of order, reduction to smaller systems, adjoint equations, adjoint systems, Green's second identity, nonhomogeneous, existence and uniqueness, maximal interval of existence, maximal or minimal solutions, initial value problems, Lipschitz conditions, generalized Lipschitz conditions, limits, asymptotic behavior of solutions, stability, asymptotic stability, Lyapunov's theorem, ordinary point, simple singularity, singular points of the first (second) kind, regular singularities, indicial equations, the Fuchsian type, self-adjoint eigenvalue problems, comparison theorems, the Picone formula, Sturm–Liouville equation, Sturm's oscillation theorems, Green's functions, the Prüfer substitution, characteristic numbers for the Sturmian system, characteristic numbers for periodic boundary conditions, stable regions, when the coefficients of the system's equation are periodic, Ascoli's lemma, Jordan canonical form, expansion and completeness theorem, Leibniz integral rule, entire function, Taylor series, interior point, the Pragmen–Lindelöf method, limit-point type, limit-circle type, Helly selection theorem, spectral functions, nondecreasing spectral matrix, point spectrum, continuous spectrum, positive semidefinite matrix, Riccati's equation, residue theorem, Picard– Lindelöf Theorem, exact differentials, Pfaffian equations, primitive functions, Natani's method, cofactor, Lagrange identity, reduced system, cluster point, index of compatibility, multiplicity, Lommel's formula, Ritz method, direct methods, method of finite differences, methodical solutions, Weber–Schafheitlin integral, Riemann surfaces, Cauchy data, Cauchy–Kowalevski theorem, complementary [transient] solution, particular [steady state] solution, electromotive force, complex impedance, resonance

We try to select the best formulation and proof for each of the following theorems.

1 Specific effective methods of solving ordinary differential equations

We will describe various methods, discuss their limitations and their reductions for specific cases. My presentation of specific methods will roughly go from the most effective methods to the least effective methods. We ignore trivial techniques.

(1) Elementary methods of integration use exact differentials to prove the integrabilty of differential equations and find their primitive funtions. Serving as a starting point for solving ordinary differential equations, these elementary methods need not use the concept of approximation.

(A) As regards to x_0, it is a consequence of the condition of integrability [Ince [18, p.17, l.13]].

Proof. Fix y_0 and change x_0 to x_1. Then the difference between the two is
$(\int_{x_1}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x,y)dy) - (\int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x,y)dy)$
$= \int_{x_1}^{x_0} P(x,y)dx + \int_{y_0}^{y} \int_{x_0}^{x_1} \frac{\partial Q}{\partial x} dxdy$
$= -\int_{x_0}^{x_1} P(x,y)dx + \int_{x_0}^{x_1} \int_{y_0}^{y} \frac{\partial P}{\partial y} dydx$
$= -\int_{x_0}^{x_1} P(x,y_0)dx.$ □

(B) The statement given in Sneddon [33, p.19, l.5–l.6] can be interpreted as Arnold [1, p.17, Theorem; Fig. 4] or Coddington–Levinson [7, p.22, Theorem 7.1; p.23, Fig. 2]. The latter interpretation is more advanced than the former one.

(C) When studying ordinary differential equations, we should aim at a higher level, partial differential equations, so that our approach to the study is directed toward that goal. The Picard– Lindelöf Theorem [Hartman [17, p.8, Theorem 1.1]] provides a large class of solvable systems of differential equations. Once we know a system of differential equations belongs to this class, we should try to find its exact solutions with the most effective method.
Methods of solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$: using exact differentials to find a two-parameter family of integral curves $u_1(x,y,z) = c_1, u_2(x,y,z) = c_2$; example: Sneddon [33, p.11, Example 2]. The solution can be found more effectively by using the formula $\frac{dx+dy}{P+Q} = \frac{dz}{R}$; example: Sneddon [33, p.12, l.8 & l.15]. The effectiveness of the method can also be improved by finding the eigenvalues of a matrix; example: Sneddon [33, p.13, Example 3]. If one of the variables is absent from one equation, then we may derive its solution easily; example: Sneddon [33, p.15, Example 4].

(D) The Pfaffian differential equations

(a) The integrability condition for a Pfaffian differential equation: Sneddon [33, p.19, Theorem 2; p.21, Theorem 5; p.34, Theorem 7; p.35, Theorem 8].
Remark. (The mathematical formulation of the second law of thermodynamics leads to a criterion for integrability of Pfaffian forms)[Sneddon [33, p.41, l.16–l.27; p.35, Theorem 8]; Zemansky–Dittman [45, p.169, (7-7); p.170, l.10–l.12; p.173, l.4–l.11]]

The algebraic criterion for integrability of Pfaffian forms [Sneddon [33, p.21, Theorem 5]] is good for calculation, while the geometric (or physical) criterion for integrability of Pfaffian forms [Sneddon [33, p.34, Theorem 7; p.35, Theorem 8]] is good for geometric (or physical) considerations. Pfaffian forms and thermodynamics are closely related. Without considering thermodynamics we cannot see the insight of Pfaffian forms; Without considering Pfaffian forms, we would have no mathematical foundation for thermodynamics. One should establish a solid connection between the two fields:
The connection from Pfaffian forms to thermodynamics: By Zemansky–Dittman [45, p.173, (7-13); p.174, (7-14)], the function μ is, apart from a multiplicative constant, a function only of the empirical temperature of the system [Sneddon [33, p.41, l.-6–l.-4]].
The connection from thermodynamics to Pfaffian forms: By Sneddon [33, p.41, l.-17–l.-16; l.-8], we see that the differential form for$đQ$ referring to a physical system of any number of independent coordinates possesses an integrating factor simply because of the second law of

thermodynamics [Zemansky–Dittman [45, p.170, l.−14–l.−12]]. By Sneddon [33, p.19, Theorem 2], a system of two independent variables has a $đQ$ which always admits an integrating factor regardless of the second law [Zemansky–Dittman [45, p.173, l.12–l.13]].
The differences between thermodynamics and the general theory of Pfaffian forms: Zemansky–Dittman [45, p.173, l.8–l.10; l.14–l.16].

(b) Solution of Pfaffian differential equation in three variables [Sneddon [33, chap. 1, §6]]

  (a) By inspection
  Remark. curl $X = (0,0,0)$ given in Sneddon [33, p.27, l.10] should have been corrected as curl $X = (0,-2x^2,6y^2)$.

  (b) Variables separable

  (c) One variable separable

  (d) Homogeneous equations

  (e) Natani's method

  (f) Reduction to an ordinary differential equation

(2) The linear equations with constant coefficients

(a) (Homogeneous) (detail: Coddington–Levinson [7, p.89, Theorem 6.5]; Method: Ince [18, p.137, l.12-l.13])
Remark. The scope of the method extends beyond the case in which the coefficients are constants: Watson–Whittaker [40, p.201, l.6–l.7; p.208, l.−11].

(b) (Nonhomogeneous)
Special cases: (Detail: Ince [18, §6.2]; method: Ince [18, p.138, l.22–l.26])
General case: Coddington–Levinson [7, p.87, (6.15); key idea: Variation of constants]

Remark 1. The method of variation of constants can be used in [Nonhomogeneous system: Coddington–Levinson [7, p.74, l.−4]; reduction of order: Coddington–Levinson [7, p.84, l.21 & p.71, l.−2–l.−1]; multiple roots: Ince [18, p.134, l.−2]; solutions with irregular singularities (Use the variation-of-constants formula to find the integral equation: Coddington–Levinson [7, p.139, (1.7); p.155, (4.23)]; then use the method of successive approximations to obtain the solutions: Coddington–Levinson [7, p.140, (1.8); p.156, (4.25)]).

Remark 2. For a linear system with constant coefficients (Hartman [17, p.57, (5.1) and $Y(0) = I$]), its solution is given by Hartman [17, p.58, (5.9) & (5.12); p.59, (5.18)]. The equality given in Hartman [17, p.61, l.−2] or Coddington–Levinson [7, p.66, l.−12] can be proved by Dowson [12, p.11, Theorem 1.19(iii); p.13, Theorem 1.21]. A fundamental matrix of a system $x' = A(t)x$, where $A(t+\omega) = A(t)$, can be represented as the product of a periodic matrix with the same period ω and a solution matrix for a system with constant coefficients (Coddington–Levinson [7, p.78, Theorem 5.1]); for the theorem's interpretations, see (Hartman [17, p.60, l.14-l.16; l.−12–l.−6]; Coddington–Levinson [7, p.80, l.8–p.81, l.6]).

(3) The Euler linear equation can be transformed into a linear equation with constant coefficients by means of the substitution $x = e^z$. See Ince [18, p.141, l.−6] and Collatz [9, chap. II, §6].

Table 1.1: Linear ODEs vs. linear systems

Linear equations of order n	Bridges	Linear systems
Homogeneous equation: Coddington–Levinson [7, p.82, l.1]; $\varphi_1(\tau) = \xi_1$	Coddington–Levinson [7, p.82, (6.2) §(6.3)]	System: Coddington–Levinson [7, p.67, (LH)]; $\hat{\varphi}(\hat{\tau}) = \hat{\xi}$
Wronskians: Coddington–Levinson [7, p.83, (6.5)]	Coddington–Levinson [7, p.82, (6.4)]	Liouville's theorem: Hartman [17, p.46, (1.5)]
Reduction of order		
Using Wronskians: Hartman [17, p.64, (8.5)] [a]		
Using a solution of the adjoint equation: Coddingto–Levinson [7, p.86, l.−4–l.−1]		
Variation of constants: Coddington–Levinson [7, p.84, l.17–l.27]		Reduction to smaller systems: Hartman [17, p.50, Lemma 3.1]; Coddington–Levinson [7, p.71, l.−2–l.−1] [b]
The Frobenius factorization: the formula given in Ince [18, p.120, l.−4] [c]		Hartman [17, p.51, Corollary 3.1]
The adjoint equation: Coddington–Levinson [7, p.84, l.−2]	Using Coddington–Levinson [p.85, (6.9)] and mathematical induction to prove thar φ_n satisfies equation given in Coddington–Levinson [7, p.85, l.−5]	The adjoint system: Hartman [17, p.62, (7.1)]
Green's second identity: Hartman [17, p.67, (8.15)] [d]		Hartman [17, p.62, (7.3)]
Nonhomogeneous equation: Coddington–Levinson [7, p.87, l.4]	Coddington–Levinson [7, p.87, l.4]	Nonhomogeneous system: Coddington–Levinson [7, p.74, l.4–l.13] [e]
Solution 1: Coddington–Levinson [7, p.87, (6.15)]		Solution: Coddington–Levinson [7, p.74, (3.1)] Key idea: Variation of constants: Coddington–Levinson [7, p.74, l.−10–l.−5]
Solution 2: Hartman [17, p.64, (8.7)] Key idea: Differentiation under the integral sign		

(4) [a] The $W(t;u,u_1,\cdots,u_{d-1})$ given in Hartman [17, p.64, (8.5)] should be replaced with $W(t;u_1,\cdots,u_{d-1},u)$. u here should be viewed as u_d in Hartman [17, p.64, (8.4)] or g in Birkhoff–Rota [3, p.36,(12)].

[b] The method of reduction to smaller systems is a modification of the variation of constants (Coddington–Levinson [7, p.71, l.−2–l.−1; the idea behind: p.84, l.17–l.27]). It is easier to recognize this point by compaing the proof of Hartman [17, p.50, Lemma 3.1] with that of Hartman [17, p.48, Theorem 2.1] than by comparing the proof of Coddington–Levinson [7, p.73, Theorem 2.5] with that of Coddington–Levinson [7, p.74, Theorem 3.1] because Hartman uses matrix motations to clarify the proof structures. The substitution $y = Z(t)z$ given in Hartman [17, p.50, l.9] corresponds with Hartman [17, p.48, (2.1)]; Hartman [17, p.50, (3.6)] corresponds with Hartman [17, p.48, (2.4)].

[c] The complete factorization of a polynomial of degree d into linear factors enables us to find all the roots by using linear factors to divide it d times. Likewise, the Frobenius factorization of an ODE of order d enables us to find all the solutions by integrating d times. The equality given in Hartman [17, p.67, (8.19)] is incorrect. The equality given in Ince [18, p.120, l.−2] can be proved as follows:

Proof. The two sides of the equality are linear differential operators of order r whose coefficient of $u^{(r)}$ is Δ_r^2 . Since $u_1,\cdots,u_r$ are solutions of the corresponding homogeneous ODEs of both operators, by Coddington–Levinson [7, p.70, l.−9], the two differential operators are equal. □

[d] Green's second identity is originated from Jackson [19, p.36, (1.35)].

[e] By the one-to-one correspondence between linear ODEs and linear systems, the existence and uniqueness of the latter solutions (Hartman [17, p.31, Corollary 5.1]) imply those of the former solutions.

(5) Riccati's equation

(a) Let $n=-2$ or $-\frac{4m}{2m+1}$, where m is zero or a positive integer. Then $\frac{dy}{dz}=az^n+by^2$ is solvable by means of algebraic, exponential, and logarithmic functions [Watson [41, §4.1–§4.12]]. However, Bernoulli's method provides the idea to find a solution, but fails to give the solution's final form. Both the trial series form and and the final form of finite terms for Euler's solution [Watson [41, §4.13]] and Cayley's solution [Watson [41, §4.14]] for w are different, but their results after calculation are the same. Schläfli's solution [Watson [41, §4.15]] establishes the relationship between Riccati's equation and Bessel's equation, but loses the ability in determining the solvability by means of algebraic, exponential, and logarithmic functions.

Remark 1. $V_1, V_2 = z\exp(\mp cz^q/q)[1\pm\frac{q+1}{q(q+1)}cz^q+\frac{(q+1)(3q+1)}{q(q+1)2q(2q+1)}c^2z^{2q}\pm\cdots]$ [Watson [41, p.89, l.14]].

Proof. Let $V(z)=\frac{v}{z}, Z=\frac{1}{z}$. Then
By Watson [41, p.89, l.10], $\frac{d^2V}{dZ^2}-c^2Z^{-2q-2}V=0$.
By Watson [41, p.88, l.−10], $\frac{v}{z}=V(z)=W(z)\exp(cZ^{-q}/(-q))$, where $W(Z)$
$=1-\frac{-q-1}{(-q)(-q-1)}cZ^{-q}+\frac{(-q-1)(-3q-1)}{(-q)(-q-1)(-2q)(-2q-1)}c^2Z^{-2q}-\frac{(-q-1)(-3q-1)(-5q-1)}{(-q)(-q-1)(-2q)(-2q-1)(-3q)(-3q-1)}c^3Z^{-3q}$
$+\cdots$ [Watson [41, p.88, l.−7–l.−6]]. □

Remark 2. The general solution of $t\frac{d^2y}{dt^2}+(a+1)\frac{dy}{dt}-y=0$ is $y=c_1F(a,t)+c_2t^{-a}F(-a,t)$ [Watson [41, p.90, l.18–l.19]].

Proof. $t\frac{d^2y}{dt^2}+(a+1)\frac{dy}{dt}-y=0$ can be transformed to the Bessel equation $z^2\frac{d^2u}{dz^2}=z\frac{du}{dz}+(z^2-a^2)u=0$ by using $u(z)=(\frac{z}{2})^a y(a,-\frac{z^2}{4})$.
By Watson [41, p.40, (8); p.90, l.17], $J_a(z)=(\frac{z}{2})^a F(a,-\frac{z^2}{4})$ [Watson [41, p.91, l.1]].
Since $J_{-a}(z)$ is also a solution of Bessel's equation, by the above transformation, we have $J_{-a}(z)=(-t)^{-a/2}y_2(t)$, where $t=-\frac{z^2}{4}$.
By Watson [41, p.91, l.1], $y_2(t)=(-t)^{-a}F(-a,t)$. □

Remark 3. $u=\frac{c_1t^{a+1}F(a+1,t)+c_2F(-a-1,t)}{c_1F(a,t)+c_2t^{-a}F(-a,t)}$ [Watson [41, p.90, l.−13]].

Proof. Since $\frac{\partial}{\partial t}F(a,t)=F(a+1,t)$,
$\frac{\partial}{\partial t}(t^{-a}F(-a,t))=(-a)t^{-a-1}F(-a,t)+t^{-a}F(-a+1,t)$.
It suffices to prove $-aF(-a,t)+tF(-a+1,t)=F(-a-1,t)$. The desired result follows from Watson [41, p.45, (1); p.91, l.1]. □

(b) Watson [41, p.91, l.10–l.18]
  (i) $\frac{du}{dt}+Au^2=Bt^m$.
  Let $u=A^{-1}d(\log y)/dt$. Then $\frac{d^2y}{dt^2}=ABt^my$.
  (ii) Let $a=1/(m+2)$ and $t=\alpha x$, where $\alpha^{1/a}AB=a^{-2}$.
  By 5(b)i, $\frac{d^2y}{dx^2}=\frac{1}{a^2}yx^{1/a-2}$.
  Let $S=\sum_{n=0}^{\infty}\frac{x^{n/a+1}}{n!(a+1)\cdots(a+n)}$. Then $\frac{d^2S}{dx^2}=\frac{1}{a^2}x^{1/a-2}S$.
  By the definition of S, Murphy's solution $y=S$ is similar to Schläfli's solution $y=F(a,t)$ [Watson [41, p.90, l.−17]].
  (iii) Let $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt$. Using repeated application of integration by parts, we have $\gamma(a,x)=\sum_{n=0}^{\infty}\frac{x^ae^{-x}x^n}{a!(a+1)\cdots(a+n)}$.
  (iv) S can be expressed as a definite integral.

Proof. Let $y = \frac{x}{2}\int_{-1}^{1} h^{-1}[\phi(h)\exp(x^{1/a}/h) + \phi(1/h)\exp(hx^{1/a})]dh$. Then

$$\begin{aligned} y &= \pi ix \operatorname{Res}(h^{-1}\phi(h)\exp(x^{1/a}/h), 0) \text{ [by the residue theorem]} \\ &= \pi iS \text{ [Take the constant term of the product of the two series using 5(b)iii].} \end{aligned}$$

□

Remark. The proof of the definite integral representation for S given in Murphy [25, p.441, l.6–p.442, l.4] is incorrect because
$\infty = \int_{-1}^{1} h^{-2}dh \neq -h^{-1}|_{-1}^{1} = -2$ [see Murphy [25, p.441, l.-1]]. Murchy should have made a small semicircular detour around $h = 0$ instead of using an integral path passing through the singularity. One can see how mathematicians had struggled with this type of problems until Cauchy submitted his paper about the residue theorem in 1831.

(c) Other things about Riccati's equation [Watson [41, §4.21]; Ince [18, §2.15]]

(6) The right attitude toward solving differential equations

In order to solve differential equations, we may want to classify differential equations by forms at first. In other words, we would like to create a table of solutions for differential equations. The more entries the table contains, the better it seems to be. In fact, this is not always true. For Bessel's equation alone, it has countless variations [Watson [41, §4.15; §4.3–§4.32]]. No advantage is to be gained by studying these variations rather than Bessel's equation. Consequently, rather than by superficial forms, we should classify differential equations by essence: ordinary points, regular singularities, and irregular singularities, as shown in Watson–Whittaker [40, p.194, l.9–p.202, l.4]. The rest of all the classifications [Coddington–Levinson [7, §4.2]] and theories [Coddington–Levinson [7, p.6, Theorem 1.2; p.10, Theorem 2.3]] of differential equations can be considered supplements or generalizations of Watson–Whittaker [40, p.194, l.9–p.202, l.4].

2 The existence and uniqueness of solutions

In the following we discuss the solutions of the initial-value problem: $x' = f(t,x)$, $x(\tau) = \xi$.

(1) Existence

(a) Coddington–Levinson [7, p.11, (2.7)] is the essence of the following four existence theorems:

(i) Suppose f is continuous on a rectangle: Coddington–Levinson [7, p.6, Theorem 1.2]. This method of using a polygonal line to string up tangent vectors cannot give all the solutions, e.g., $x' = x^{1/3}$; Coddington–Levinson [7, p.7, l.-18–l.-8]. For approximation, Coddington–Levinson [7, p.3, Theorem 1.1] is sufficient. If the solution is unique, it is unnecessary (Coddington–Levinson [7, p.7, l.11–l.19]) to use Ascoli's lemma (Coddington–Levinson [7, p.5, l.-13–l.-12]).

(ii) Uniform Lipschitz continuity (Coddington–Levinson [7, p.10, Theorem 2.3])

(iii) The method of successive approximation (Coddington–Levinson [7, p.12, Theorem 3.1]; Hartman [17, p.8, Theorem 1.1])

Remark 1. The method given in proof of Coddington–Levinson [7, p.10, Theorem 2.3] is more effective than that given in the proof of Coddington–Levinson [7, p.12, Theorem 3.1]. This is because Coddington–Levinson [7, p.11, (2.7)] provides a faster convergence than the inequality given in Coddington–Levinson [7, p.13, l.−10–l.−9].

Remark 2. The proof of uniqueness given in Coddington–Levinson [7, p.13, l.15] is better than that given in Hartman [17, p.9, l.−5–p.10, l.2] because the latter uses mathematical induction, an infinite procedure.

(iv) The linear systems: Pontryagin [27, p.167, l.−9–l.−3]
Remark. The proof of Hartman [17, p.31, Corollary 5.1] first finds a small interval of existence, and then uses mathematical induction to expand the domain step by step. From a theoretical viewpoint, the proof is stuck to formality. In contrast, the method given in the proof of Pontryagin [27, p.167, l.−9–l.−3] is more flexible and effective. From practical viewpoint, the computations based on Hartman's proof would be disastrous.

(b) Maximal intervals: Hartman [17, pp.12–13, Theorem 3.1]

Remark 1. In one word, the proof of Hartman [17, pp.12–13, Theorem 3.1] is based on Hartman [17, p.11, Corollary 2.1] and mathematical induction.

Remark 2. Hartman [17, pp.12–13, Theorem 3.1] gives the extent of domain expansion, while Coddington–Levinson [7, p.15, Theorem 4.1] fails to do so.

Remark 3. (A good theorem should provide complete information) If we treat $\mathbb{R}^2$ as a topological subspace of its one-point compactification S^2 and denote the boundary relative to S^2 as ∂_∞, then the geometric meaning of "$y(t)$ tends to ∂E as $t \to \omega+$" given in Hartman [17, p.13, l.5–l.7] is "$(t, y(t))$ tends to $\partial_\infty E$ as $t \to \omega+$". A good theorem should provide complete information. In the above sense, the conclusion of Hartman [17, pp.12–13, Theorem 3.1] gives a complete geometric picture, while the result given in Hirsch–Smale–Devaney [11, p.398, l.−10–l.−9] to which Hirsch–Smale–Devaney [11, p.398, Theorem] leads fails to completely describe what it should.

Remark 4.

Lemma. Let $f(t,y)$ be continuous on a (t,y)-set E. Let $y = y(t)$ be a solution of $y' = f(t,y)$ on $[a,\delta)$, $\delta < \infty$, for which $\exists t_n \in [a,\delta) : (\lim_{n\to\infty} t_n = \delta$ and $\lim_{n\to\infty} y(t_n) = y_0)$. If $f(t,y)$ is bounded on the intersection of E and a vicinity of the point (δ, y_0), then $\lim_{t\to\delta} y(t) = y_0$ [Hartman [17, p.13, Lemma 3.1]].

Proof. I. By hypothesis, we may take a small $\varepsilon > 0$, and a large $M_\varepsilon > 0$ such that $|f(t,y)| \leq M_\varepsilon$ for $(t,y) \in E \cap \{(t,y) | 0 \leq \delta - t \leq \varepsilon, |y - y_0| \leq \varepsilon\}$.
II. Take a large n such that $0 < \delta - t_n \leq \frac{\varepsilon}{2M_\varepsilon}$ and $|y(t_n) - y_0| \leq \varepsilon/2$. Then
III. $\forall t_{t_n \leq t < \delta}, |y(t) - y(t_n)| < M_\varepsilon(\delta - t_n)$.

Proof. Assume that III were false. Then
$\exists t^*_{t_n \leq t^* < \delta} : |y(t) - y(t_n)| \geq M_\varepsilon(\delta - t_n)$.
Let $t^1 = \min\{t \in [a,\delta) : |y(t) - y(t_n)| = M_\varepsilon(\delta - t_n)\}$. Then
1. $t_n < t^1 < \delta$.
2.

$$\begin{aligned} |y(t^1) - y(t_n)| &= M_\varepsilon(\delta - t_n) \\ &\leq \varepsilon/2 \quad \text{(by II)}. \end{aligned}$$

3.

$$\begin{aligned}\forall t_{t_n \le t < \delta}, |y(t) - y_0| &\le |y(t) - y(t_n)| + |y(t_n) - y_0| \\ &< M_\varepsilon(\delta - t_n) + |y(t_n) - y_0| \quad \text{(by the definition of } t^1\text{)} \\ &\le \varepsilon \quad \text{(by II).}\end{aligned}$$

4. $\forall t_{t_n \le t < \delta}, |y'(t)| = |f(t, y(t))| \le M_\varepsilon$.

Proof.

$$\begin{aligned}\delta - t \le \delta - t_n &\le \frac{\varepsilon}{2M_\varepsilon} \quad \text{(by II)} \\ &\le \varepsilon.\end{aligned}$$

By 3, $|y(t) - y_0| \le \varepsilon$. The result follows from I. □

5.

$$\begin{aligned}|y(t^1) - y(t_n)| &\le M_\varepsilon(t^1 - t_n) \quad \text{(by 4)} \\ &< M_\varepsilon(\delta - t_n) \quad \text{(by 1).}\end{aligned}$$

This would contradict the definition of t^1. □

□

Remark. The proof of Hartman [17, p.13, Lemma 3.1] is hard to read because all it contains is a series of formulas with little documentation.

Remark 5. $y(t)$ tends to the boundary ∂E of E as $t \to \omega+$ [Hartman [17, p.13, l.2–l.3]].

Proof. I. Because $(b_k, y(b_k)) \notin \bar{E}_{n(k)}$, $(b_1, y(b_1)), (b_2, y(b_2)), \cdots$ is either unbounded or has a cluster point on the boundary ∂E of E [Hartman [17, p.13, l.−12–l.−11]].
II. Assume the statement "$y(t)$ tends to the boundary ∂E of E as $t \to \omega+$" were false. Then
$\exists t_n \in [a, \omega+) : \lim_{n\to\infty}(t_n, y(t_n)) = (\omega+, y_0) \in \bar{E}_m$.
1. Consequently, f is bounded on the intersection of E and a vincinity of the point $(\omega+, y_0)$. That is,
$\exists c \in [a, \omega+), M > 0 : (c \le t < \omega+) \Rightarrow |y'(t)| = |f(t, y(t)| \le M$. Thus,
$y(t)$ is uniformly continuous on $[c, \omega)$. We may define $y(\omega+) = \lim_{t\to\omega+} y(t)$. By Dugundji [13, p.302, Theorem 5.2], the extension of $y(t)$ is uniformly continuous on $[a, \omega+]$.
2. $y(t) : [a, \omega+] \to \mathbb{R}$ is differentiable at $\omega+$ and is a solution of $y'(t) = f(t, y(t))$ on $[a, \omega+]$.

Proof.

$$\begin{aligned}y(t) &= y(a) + \lim_{t\to\omega+} \int_a^t y'(s)ds \quad \text{(by definition)} \\ &= y(a) + \lim_{t\to\omega+} \int_a^t f(s, y(s))ds \\ &= y(a) + \int_a^{\omega+} f(s, y(s))ds \quad \text{(Rudin [31, p.27, Theorem 1.34]).}\end{aligned}$$

Consequently, $\forall t \in [a, \omega+], y(t) = y(a) + \int_a^t f(s, y(s))ds$ and $y'(t) = f(t, y(t))$. □

3. Since dist $(\bar{E}_m, \partial E) \geq 1/m$, by Hartman [17, p.11, Corollary 2.1], there exists a $\delta > \omega+$ such that the solution $y(t)$ on $[a, \omega+]$ can be extended to $[a, \delta]$. This would contradict the fact that $[a, \omega+)$ is the right maximal interval. □

(c) Maximal or minimal solutions: Hartman [17, p.25, Lemma 2.1]

Remark 1. The definition $\Phi = \sup|\varphi|$ given in Coddington–Levinson [7, p.46, l.5] should be replaced with $\Phi = \sup\varphi$. Even with this correction, the definition Φ is still problematic because there exists an initial problem (**E**) [Coddington–Levinson [7, p.42, (1.1)]] that we know only some of its solutions, but not all of them. In such a case, we may find a supremum of some solutions, but we cannot find the supremum of all solutions. Someone may argue that the shortcoming is amended by proving $\Phi = \lim_{x\to\infty} \varphi_{1/m}$ (Coddington–Levinson [7, p.47, l.2–l.3]). However, $\varphi_{1/m}$ is not accessible because φ_e is defined by means of $\varphi_i (i = 0, 1, \ldots, n-1)$ and φ_i's depend on Φ which is not accessible. The construction given in Hartman [17, p.25, (2.4)] enables us to avoid considering all the solutions.

Remark 2. Hartman [17, p.26, Theorem 4.1] can be used to find maximal intervals of existence (Hartman [17, p.29, l.−4–l.−3]). The proof of Hartman [17, p.29, Theorem 5.1] supplies an estimate for solutions of Hartman [17, p.30, (5.3)].

(2) Uniqueness

(a) The Lipschitz condition: Coddington–Levinson [7, p.10, Theorem 2.2]

(b) The generalized Lipschitz condition for continuous f : Coddington–Levinson [7, pp.48–49, Theorem 2.1]

Remark 1. Coddington–Levinson [7, p.49, l.12–l.19]

Remark 2. We must assume $\psi(t, 0) = 0$; this assumption is used in Coddington–Levinson [7, p.51, l.7]. It is unnecessary to assume that $\psi(t, r)$ is nondecresing in r for fixed t. The proof of this theorem is the same as that of Coddington–Levinson [7, p.49, Theorem 2.2] except the part given in CoddingtonLevinson [7, p.50, l.−17–p.51, l.2]. We may use the method of proving Lakshmikantham [23, p.163, (6)] to prove Coddington–Levinson [7, p.50, (2.9)]. The equality given in Lakshmikantham [23, p.163, l.−5] follows from Coddington–Levinson [7, p.29, Theorem 7.4]. Note that the proof of Hartman [17, p.31, Theorem 6.1] is incorrect because there is a minus sign on the right-hand side of the equality in Hartman [17, p.27, (4.5)], while there is no minus sign on the right-hand side of the equality in Hartman [17, p.32, l.18].

(c) The generalized Lipschitz condition for general f : Coddington–Levinson [7, p.49, Theorem 2.2]

Remark 1. We must assume $\psi(t, 0) = 0$; this assumption is used in Coddington–Levinson [7, p.51, l.7].

Remark 2. At first glance, Coddington–Levinson [7, p.51, l.−12–l.−8] seems to say that the solution in Coddington–Levinson [7, p.43, Theorem 1.1] is unique under the hypothesis of Coddington–Levinson [7, p.43, Theorem 1.1]. Actually, it discusses how we generalize Coddington–Levinson [7, p.49, Theorem 2.2]:

Suppose the f given in Coddington–Levinson [7, p.49, Theorem 2.2] merely satisfies the Carathéodory hypothesis. Then the conclusion of Coddington–Levinson [7, p.49, Theorem 2.2] is still valid if the conditions about ψ are satisfied.

Thus, if a theorem's hypothesis is too complicated, not only the formulation is inconvient for application, but also its generalization may easily be misinterpreted. Combining Coddington–Levinson [7, p.49, Theorem 2.2] with the specification given in Coddington–Levinson [7, p.49, l.13], we may simplify the hypothesis of the uniqueness theorem as follows:

Let f be defined in $R = \{(t,x) | |t-\tau| \le a, |x-\xi| \le b; a,b > 0\}$ and suppose it is measurable in t for each fixed x; continuous in x for each fixed t. If for every (t,x_1), (t,x_2) in R, $|f(t,x_1) - f(t,x_2)| \le k|x_1 - x_2|$, then there exists at most one absolutely continuous solution φ of (E) (Coddington–Levinson [7, p.42, l.6]) in R on $|t-\tau| \le a$ for which $\varphi(\tau) = \xi$.

3 Limits concerning ODEs

3.1 Let n be the index of a sequence of ODEs and then let $n \longrightarrow +\infty$

(1) Hartman [17, p.4, Theorem 2.4]

(2) Maximal interval of existence: Hartman [17, p.14, Theorem 3.2]

(3) Linear systems: Hartman [17, p.55, Corollary 4.1]

3.2 Asymptotic behavior of solutions of a linear system $x' = (A + V(t) + R(t))x$ as $t \longrightarrow \infty$

Theorem 3.1. Coddington–Levinson [7, pp.92–93, Theorem 8.1]

Example 3.1. Coddington–Levinson [7, p.91, l.-4–p.92, l.15]

Remark. The only thing that requires a proof is the statement given in Coddington–Levinson [7, p.92, l.15].

Proof. $\int_0^t (1+\tau^{-\alpha})^{1/2} d\tau = \int_0^t (1+O(\tau^{-\alpha}))d\tau$
$= t + c + O(t^{1-\alpha}) = t(1 + O(t^{-1}) + O(t^{-\alpha}))$. □

The same argument can be used to prove the formula given in Bucur [6, p.57, l.15], but the above proof does not apply to the case $\alpha = 1$. Bucur [6, p.58, (7)] shows that the asymptotic behaviour of the solutions of a differential equation cannot always be deduced from the behaviour of the solutions of a limiting equation.

3.3 Stability

Theorem 3.2. (**Lyapunov's theorem**) Pontryagin [27, pp.208–211, Theorem 19]; Coddington–Levinson [7, p.314, l.-10–p.315, l.21]

Remark 1. The solution to the problem given in Coddington–Levinson [7, p.315, l.16] can be found in Wikipedia [43, §Integral form for continuous functions, (b)].

Remark 2. In Coddington–Levinson [7, p.315, (1.4)], σ can be taken as μ given in Coddington–Levinson [7, p.315, (1.5)]. See Coddington–Levinson [7, p.316, l.3–l.4]. In contrast, the "α" given in Pontryagin [27, p.211, l.-11] is not accessible. If we trace its construction (Pontryagin [27, p.210, l.13; p.208, l.2; p.206, l.4–l.5]), we find that its existence is provided by reduction to absurdity (Rudin [29, p.31, l.12–l.16]). The proof pattern of Pontryagin [27, p.208, Theorem 19] follows that of Hartman [17, p.40, Theorem 8.4]. Since the latter theorem is more general, its derived constants are usually less effective. Another problem with the proof of Pontryagin [27, 208, Theorem 19] is that the W in Pontryagin [27, p.207, (20)] depends on ψ_i, which in turn depends on Pontryagin [27, p.206, (16)]. In other words, μ, ν in Pontryagin [27, p.206, (13)], and hence α in Pontryagin [27, p.210, l.14] depend on Pontryagin [27, p.206, (16)]. Thus, in order to determine α in Pontryagin [27, p.210, l.14], we must actually solve Pontryagin [27, p.206, (16)]. However, our goal is to determine a solution's stability without actually solving differential equations.

Remark 3. Read Wang [38, §6.3.1].

4 Singularities

Table 4.1: Singularities

Linear equations of order n	Linear systems
	The standard form of fundamental matrices: Hartman [17, pp.70–71, Theorem 10.1]
Singularities of the first kind [a]	
Form: Coddington–Levinson [7, p.122, (5.1) & (5.2)]	Hartman [17, p.73, (11.1) & (11.2)]
The "if" portion of Hartman [17, p.85, Theorem 12.1]	Coddington–Levinson [7, p.111, Theorem 2.1] [b, c]
The "only if" portion of Hartman [17, p.85, Theorem 12.1] [b, c]	Hartman [17, p.74, Theorem 11.2]
	Formal power series solutions converge (Hartman [17, p.78, Theorem 11.3]) [f]
	Solutions based on the roots of indicial equation (Hartman [17, p.85, l.9–l.14]): Hartman [17, pp.81–82, Theorem 11.4]
The Fuchsian type	
Coddington–Levinson [7, pp.129–130, Theorem 6.4]	Coddington–Levinson [7, p.129, Theorem 6.3]
Singularities of the second kind	
The solution of Coddington–Levinson [7, p.138, (1.2)] can be obtained from the first component of the solution of Coddington–Levinson [7, p.169, (7.4)] through the transformation given in Coddington–Levinson [7, p.169, (7.2) & (7.3)] [g]	The formal solutions exist (if all the characteristic roots of A_0 are simple: Coddington–Levinson [7, pp.142–143, Theorem 2.1]; if A_0 has multiple characteristic roots: Coddington–Levinson [7, p.168, Theorem 6.1]) and corresponding to the formal solutions actual solutions exist which have the formal solutions as *asymptotic expansions* (if all the characteristic roots of A_0 are simple: Coddington–Levinson [7, pp.160–161, Theorem 4.1] [real domain]; Coddington–Levinson [7, p.163, Theorem 5.1] [complex domain] [h]; if A_0 has multiple characteristic roots: transform the multiple case into the simple case, see Coddington–Levinson [7, p.167, l.−15–p.168, l.2]). Sketch of the method: Using the variation-of-constants formula; Coddington–Levinson [7, p.152, l.9–l.−6; p.155, (4.23)]
	The number of linear independenct solutions: Hartman [17, p.87, Theorem 13.1]

[a] A singular point of the first kind in Coddington–Levinson [7, p.111, l.21] = a simple singularity in Hartman [17, p.73, l.20].

[b] The meanings of Hartman [17, p.73, Corollary 11.1; p.79, Corollary 11.2] are given by Coddington–Levinson [7, p.118, l.−9–p.119, l.2].

[c] The statement given in Hartman [17, p.79, l.18–l.19] can be proved using the formula given in Coddington–Levinson [7, p.91, l.14].

[d] (The primitive model with a simple setting) If one wants to prove the statement given in Hartman [17, p.85, l.−3–l.−2] or that given in Coddington–Levinson [7, p.125, l.11–l.14], there are three ways to do so: Read Yosida [44, p.37, l.−1–p.39, l.16], the proof of Hartman [17, pp.70–71, Theorem 10.1], or the proof of Coddington–Levinson [7, p.109, Theorem 1.1]. The first way provides the primitive model for the case $d = 2$. One can easily generalize the argument to the general case of order d. This is the reason that the first way is more inspiring and insightful than the last two ways. The Jordan canonical form and ODEs of order d involved in the last two ways are unnecessary complications for our pursuit. Note that for the Jordan canonical form, Hartman [17, p.59, (5.15)] is *consistent* with Jacobson [20, vol.2, p.97, (29)] through a proper order of the elements of the basis, while Coddington–Levinson [7, p.63, l.−5] is *not*. *Consequently*, in order to obtain the solutions of desired form given in Hartman [17, p.85, l.−2], we may consider the fundamental matrices $Y(t)T$ given in Hartman [17, p.71, l.8], $Y(t)$ given in Hartman [17, p.71, (10.2)], or $S(z-z_0)^P T$ given in Coddington–Levinson [7, p.110, l.5], but not $S(z-z_0)^P$.

[e] For the proof of the "only if" portion of Hartman [17, p.85, Theorem 12.1], there is a gap in Hartman [17, p.84, l.−14] because Hartman failed to prove that the solutions of Hartman [17, p.86, (12.9)] inherited from the $d-1$ linearly independent solutions of Hartman [17, p.84, (12.1)] are linearly independent. See Coddington–Levinson [7, p.126, l.12–l.16].

[f] (Formal power power series vs. formal Laurent series) Through the transformation given in Hartman [17, p.79, (11.20)], we may consider Hartman [17, p.78, Theorem 11.3] instead of Coddington–Levinson [7, p.117, Theorem 3.1; p.116, l.12] without loss of generality (Hartman [17, p.80, l.14]).

[g] When $r = 1$, Coddington–Levinson [7, p.169, (7.1) & (7.2)] at $z = \infty$ correspond to Hartman [17, p.85, (12.5) & (12.2)] at $t = 0$. When $r = 0$ at $t = 0$, the latter transformation reduces to "$L_n = 0$ (Coddington–Levinson [7, p.81, l.−4]) $\xrightarrow{Coddington-Levinson[7,p.82,(6.3)]}$ Coddington–Levinson [7, p.82, (6.1) & (6.2)]".

[h] (Real domain $\xrightarrow{the\ Pragmen-Lindel\ddot{o}f\ method}$ complex domain) For the discussion of the method, read Wang [38, §4.2 (1), (2) & (3)].

5 Oscillation theory

5.1 Comparison theorems

Theorem 5.1. (The first comparison theorem)(Ince [18, §10.31]; Nagle–Saff–Snider [26, p.704, Theorem 17; p.713, group project C])

Let ϕ_1 be a nontrivial solution to the Sturm–Liouville equation
$\frac{d}{dx}[p_1(dy/dx)] + Q_1 y = 0, a < x < b,$
and let ϕ_2 be a nontrivial solution to
$\frac{d}{dx}[p_2(dy/dx)] + Q_2 y = 0, a < x < b.$
Assume that $p_1 \geq p_2 > 0$ and $Q_1 \leq Q_2$ for x in $[a,b]$. Then between any two consecutive zeros x_1 and x_2 of ϕ_1 in $[a,b]$, there is a zero of ϕ_2, unless ϕ_1 and ϕ_2 are linearly dependent on $[x_1,x_2]$, in which case $Q_1(x) \equiv Q_2(x)$ on $[x_1,x_2]$.

Proof. (a) (The Picone formula)
Let $\phi_2 \neq 0$ in $[x_1,x_2]$. By Ince [18, p.226, l.9–l.10],
$\int_{x_1}^{x_2}(Q_2 - Q_1)\phi_1^2 \mathrm{d}x + \int_{x_1}^{x_2}(p_1 - p_2)(\phi_1')^2 \mathrm{d}x + \int_{x_1}^{x_2} p_2(\phi_1' - \frac{\phi_1\phi_2'}{\phi_2})^2 \mathrm{d}x = 0.$

(b) If $\phi_2 \neq 0$ in (x_1,x_2), but ϕ_2 is 0 at $x = x_1$ or $x = x_2$,
then the Picone formula still holds (Ince [18, p.226, l.-21–l.-20]).

(c) If $Q_1(x) \not\equiv Q_2(x)$ on $[x_1,x_2]$, then ϕ_2 must have a zero in (x_1,x_2). Otherwise, the left-hand side of the Picone formula will be greater than 0.

(d) If $Q_1(x) \equiv Q_2(x)$ on $[x_1,x_2]$, then ϕ_1 and ϕ_2 are linearly dependent on $[x_1,x_2]$.
Proof. In an interval where $p_1(x) \equiv p_2(x)$, ϕ_1 and ϕ_2 satisfy the same differential equation and their Wronskian is 0.
In an interval where $p_1(x) \neq p_2(x)$, $\phi_1' = \phi_2' = 0$. Consequently, their Wronskian is also 0.

□

Remark. Theorem 5.1 is the key point of the oscillation theory, but it is a *static* formulation. The beautiful part of the theory lies in the continuous and *dynamic* formulation given in Ince [18, p.229, l.16–l.25]. The latter formulation is an application of the Sturm's first comparison theorem given by Ince [18, p.228, l.-25–l.-6].

Theorem 5.2. (The second comparison theorem)(Ince [18, p.229, l.-16–l.-14])

5.2 Sturm's oscillation theorems

For the existence of eigenvalues, the method given in Ince [18, pp.231–235, §10.6] are more effective than the method given in Coddington–Levinson [7, p.195, l.15–p.197, l.8] (self-adjoint eigenvalue problems of the nth order). This is because the latter method *unnecessarily* uses the complicated Ascoli theorem (Coddington–Levinson [7, p.195, l.-8]) and the *inaccessible* concept of supremum (e.g., Coddington–Levinson [7, p.196, l.-14]) to find eigenvalues. By comparing the method given in Ince [18, pp.231–235, §10.6] with the method given in Birkhoff–Rota [3, chap. 10, §6–§8] (self-adjoint eigenvalue problems of the

2nd order), we find that the latter method creates no new ideas. However, by using the Prüfer substitution (Birkhoff–Rota [3, chap. 10, §5]), the latter method does make the proofs more rigorous. In order to fully understand Birkhoff–Rota [3, chap. 10, §6–§8], the following three supplements should be added:

(1) The statement that $\sin\theta(t)$ can vanish only at isolated points (Birkhoff–Rota [3, p.268, l.14]) can be proved as follows:

Proof. Assume there exists a sequence of t_n such that $\sin\theta(t_n) = 0$ and $t_n \to t_0$ as $n \to +\infty$.
Then $\frac{d\sin\theta(t)}{dt}|_{t=t_0} = 0$.
However, $\frac{d\sin\theta(t)}{dt}|_{t=t_0} = \cos\theta(t_0)\theta'(t_0) = \theta'(t_0) > 0$ (Birkhoff–Rota [3, p.267, (22)]). □

(2) Replace the statements given in Birkhoff–Rota [3, p.270, l.12–l.14] by the formula given in Coddington–Levinson [7, p.212, l.-12].

(3) Replace the statements given in Birkhoff–Rota [3, p.273, l.-20–l.-18] by the statements given in Coddington–Levinson [7, p.213, l.-17–l.-16].

Common assumptions for the theorems in this subsection: Ince [18, p.231, l.26–l.30; l.-12–l.-10; l.-6; p.232, l.24–l.25]

Theorem 5.3. (Ince [18, p.232, l.-18–l.-15])
An extra assumption is required: Ince [18, p.232, l.2–l.3].

Remark. The fact that $\mu_i, \mu_{i+1}, \mu_{i+2}, \ldots$ have the limit-point Λ_2 (Ince [18, p.232, l.9–l.10]) can be proved by Nagle–Saff–Snider [26, p.699, Theorem 13].

Theorem 5.4. (Ince [18, p.233, l.-19–l.-14])
An extra assumption is required: Ince [18, p.232, l.-12].

Theorem 5.5. (Ince [18, p.235, l.9–l.15])
Extra assumptions are required: Ince [18, p.233, l.-5–p.234, l.3].

5.2.1 Application to the Sturm–Liouville system

Theorem 5.6.
Assumptions: Ince [18, p.235, l.23–l.25].
Conclusions: Ince [18, p.235, l.-16–l.-13].

Theorem 5.7. (Ince [18, p.237, l.4–l.14])
An extra assumption is required: the case given in Ince [18, p.236, l.-8] is excluded.

Proof. Let $|\lambda|$ be our new λ. Then apply Ince [18, p.235, Theorem III] to the two cases given in Ince [18, p.236, l.2–l.3]. □

Remark 1. For the excluded case, the theorem is still valid (Bôcher [4, p.7, l.-23–l.-13]), except that the following modifications for λ_0^+, λ_0^- are required: Bôcher [4, p.9, l.4–l.10].

Remark 2. For the conditions given in Ince [18, p.236, l.15–l.18], we take one sub-interval in the case $\lambda > 0$ and take a different sub-interval in the case $\lambda < 0$.

6 Boundary value problems

6.1 Green's function in one dimension

The definition of a Green's function is given by Ince [18, p.254, l.16–l.21]. The uniqueness of Green's function follows from Ince [18, p.254, l.19–l.20] (Coddington–Levinson [7, p.192, l.23–l.27]). Indeed, as a function of x, $G-\tilde{G}$ is of class C^{n-1} because G and $\tilde{G}$ have the same discontinuity at $x=\xi$. Although Green's function is unique, there are many methods for its construction. Here are some examples: Ince [18, p.254, l.-19–p.255, l.-11], Coddington–Levinson [7, p.190, l.28–p.192, l.1] (let $l=0$), Birkhoff–Rota [3, p.286, (67)]. The first example provides a solution which proposes a plan but fails to execute it for the treatment of both the differential equation and the boundary conditions. The second example provides a solution which finishes the treatment of the differential equation, but only proposes a plan for dealing with the boundary conditions without finishing the plan. The third example provides a solution which finishes the treatment of both the differential equation and the boundary conditions. The more precise the form is, the stroger properties of Green's function we may obtain from it (Compare Ince [18, p.254, l.16–l.21] with Coddington–Levinson [7, p.192, l.12–l.20]). The property given in Coddington–Levinson [7, p.192, l.14–l.15] can be used to prove the formula given in Ince [18, p.257, l.2] (Rudin [31, p.27, Theorem 1.34]). The reason given in Ince [18, p.257, l.1] is incorrect. Now we discuss the above three examples in detail.

(1) Corrections for the first example:

(a) "$P(G,H)=0$ when $x=a$ and when $x=b$" given in Ince [18, p.256, l.7–l.8] should be replaced with "$P(G,H)|_a^b=0$ (Ince [18, p.213, l.19])".

(b) $p_0[H\frac{d^{n-1}G}{dx^{n-1}}-G\frac{d^{n-1}H}{dx^{n-1}}]$ given in Ince [18, p.256, l.12] should be replaced with $p_0[H\frac{d^{n-1}G}{dx^{n-1}}+(-1)^{n-1}G\frac{d^{n-1}H}{dx^{n-1}}]$.

(c) $p_0(\xi_1)H(\xi_1,\xi_2)\lim[\frac{d^{n-1}G}{dx^{n-1}}]_{\xi_1-\varepsilon}^{\xi_1+\varepsilon}-p_0(\xi_2)G(\xi_2,\xi_1)\lim[\frac{d^{n-1}H}{dx^{n-1}}]_{\xi_2-\varepsilon}^{\xi_2+\varepsilon}=0$ should be relplaced with $p_0(\xi_1)H(\xi_1,\xi_2)\lim[\frac{d^{n-1}G}{dx^{n-1}}]_{\xi_1-\varepsilon}^{\xi_1+\varepsilon}+(-1)^{n-1}p_0(\xi_2)G(\xi_2,\xi_1)\lim[\frac{d^{n-1}H}{dx^{n-1}}]_{\xi_2-\varepsilon}^{\xi_2+\varepsilon}=0.$

(d) "$p_0(\xi_1)\lim[\frac{d^{n-1}G}{dx^{n-1}}]_{\xi_1-\varepsilon}^{\xi_1+\varepsilon}=p_0(\xi_2)\lim[\frac{d^{n-1}H}{dx^{n-1}}]_{\xi_2-\varepsilon}^{\xi_2+\varepsilon}=1$" should be replaced with "$p_0(\xi_1)\lim[\frac{d^{n-1}G}{dx^{n-1}}]_{\xi_1-\varepsilon}^{\xi_1+\varepsilon}=(-1)^n p_0(\xi_2)\lim[\frac{d^{n-1}H}{dx^{n-1}}]_{\xi_2-\varepsilon}^{\xi_2+\varepsilon}=1$".

(2) Supplements of the second example: $Lu=lu+f$ [Coddington–Levinson [7, p.191, l.15]] follows from Coddington–Levinson [7, p.87, Theorem 6.4] with $\tau=a$ and $\hat{\xi}=\hat{0}$. By Kaplan [22, p.255, Theorem], $u(t,l)$ is of class C^n in t [Coddington–Levinson [7, p.191, l.14–l.15]].

(3) Supplements of the third example: For a natural and smooth construction, see Hartman [17, pp.327–328, (x)]. Bernd [2] provides the motivation of the construction of Green's function given in Birkhoff–Rota [3, p.286, (67)]. Bernd [2, Example 3] relates Green's function to the Dirac delta function and the Heaviside function. Bernd [2, (5.35)] motivates us to generalize the relationships to the abstract level of functional analysis (Rudin [30, p.206, Exercise 10; p.378, l.-6]). Compare Bernd [2, (5.27)] with the formula given in Rudin [30, p.206, l.9].

Remark 1. For the physical motive of the construction, read Marion–Thornton [24, §3.9].
Remark 2. Green's function in three dimensions: physical meaning and plan of construction [Jackson [19, p.38, l.-13–p.40, l.3; p.40, l.17–l.31]]; execution in detail: [Jackson [19, p.58, l.3–p.59, l.9; p.64, l.9–l.20]]. The requirement of Jackson [19, p.126, (3.146)] reveals that the seemingly artifical Jackson [19,

p.125, (3.143)] is a natural consequence of the symmetry of the 3-dim Green function [Jackson [19, p.125, (3.142)]].

Theorem 6.1. *A boundary value problem is self-adjoint if and only if its Green function $G(t,\tau)$ satisfies $G(t,\tau)=\bar{G}(\tau,t)$.*

Proof. (a) $\Rightarrow$: Coddington–Levinson [7, p.193, l.13–l.20].

(b) $\Leftarrow$: $G(t,\tau)=\bar{G}(\tau,t)$ implies Coddington–Levinson [7, p.193, (2.9)]. Then read Coddington–Levinson [7, p.193, l.21–l.24].

□

6.2 Green's function in three dimensions

$G(\mathbf{x},\mathbf{x}')=\frac{16\pi}{ab}\sum_{l,m=1}^{\infty}\sin(\frac{l\pi x}{a})\sin(\frac{l\pi x'}{a})\sin(\frac{m\pi y}{b})\sin(\frac{l\pi y'}{b})\frac{\sinh(K_{lm}z_<)\sinh[c-K_{lm}z_>]}{K_{lm}\sinh(K_{lm}c)}$, where $K_{lm}=\pi(l^2/a^2+m^2/b^2)^{1/2}$ [Jackson [19, p.129, (3.168)]].

Proof. I. Let $G(\mathbf{x},\mathbf{x}')=\frac{16\pi}{ab}\sum_{l,m=1}^{\infty}\sin(\frac{l\pi x}{a})\sin(\frac{l\pi x'}{a})\sin(\frac{m\pi y}{b})\sin(\frac{l\pi y'}{b})g(l,m,z,z')$ (by symmetry and a theorem similar to Coddington–Levinson [7, p.197, Theorem 4.1]).
$\nabla_{\mathbf{x}}^2 G=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$
$=\frac{16\pi}{ab}\sum_{l,m=1}^{\infty}\sin(\frac{l\pi x}{a})\sin(\frac{l\pi x'}{a})\sin(\frac{m\pi y}{b})\sin(\frac{l\pi y'}{b})[\frac{\partial^2 g}{\partial z^2}-(\frac{l^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2})g]$.
$-4\pi\delta(\mathbf{x}-\mathbf{x}')$
$=-4\pi\delta(x-x')\delta(y-y')\delta(z-z')$ [Cohen-Tannoudji–Diu–Laloë [8, vol. 2, p.1477, (59)]]
$=-4\pi\delta(z-z')\sum_{l,m=1}^{\infty}\frac{4}{ab}\sin(\frac{l\pi x}{a})\sin(\frac{l\pi x'}{a})\sin(\frac{m\pi y}{b})\sin(\frac{l\pi y'}{b})$ [Cohen-Tannoudji–Diu–Laloë [8, vol. 1, p.100, (A-32)]].
Because $\nabla_{\mathbf{x}}^2 G=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ [Jackson [19, p.120, (3.116)]] and $\{\sin(\frac{l\pi x}{a})\sin(\frac{m\pi y}{b})\}_{lm}$ are linearly independent,
$\frac{\partial^2 g}{\partial z^2}-K_{lm}^2 g=-\delta(z-z')$.
II. The desired result follows from Birkhoff–Rota [3, p.286, (67)]. □

Remark. (The general method of finding a Green function's eigenfunction expansion: using symmetry)

In order to reduce the problem of finding a 3-dim Green function to the problem of finding 1-dim Green function, we should summarize the proof of Jackson [19, p.121, l.2, (3.120)], the proof of Jackson [19, p.125, (3.141)], and Part I of the above proof as follows: Put the unit charge into the volume of interest. Let $\mathbf{x}'$ be its position. Let x,y,z be the Green function's three variables. Now use z' to divide the volume into two regions: I. $\{\mathbf{x}|z<z'\}$; II. $\{\mathbf{x}|z>z'\}$. In these two regions, the Poisson equation $\nabla_{\mathbf{x}}^2 G=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ is reduced to the Laplace equation $\nabla_{\mathbf{x}}^2 G=0$. Let $\{\phi_{lm}(x,y)\}_{lm}$ be the basis of the solution space. By symmetry and a theorem similar to Coddington–Levinson [7, p.197, Theorem 4.1], we have
$G(\mathbf{x},\mathbf{x}')=\sum_{lm}g_{lm}(z,z')\phi_{lm}(x,y)\phi_{lm}(x',y')$. By substituting this expression for G into $\nabla_{\mathbf{x}}^2 G=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ and using Cohen-Tannoudji–Diu–Laloë [8, vol. 1, p.100, (A-32)], we will obtain the equation for 1-dim Green function.

In the last paragraph, we have used the fact that G is symmetric in (x,y) and (x',y'). In order to find the solutions of the equation for the 1-dim Green function, we should use the fact that G is symmetric in z and z'. This usage of symmetry is more subtle, more refine, and more interesting than the previous one. In view of the example given in Coddington–Levinson [7, p.222, l.9–l.14], the algebraic methods of sloving the boundary value problems such as Jackson [19, p.121, l.2, (3.120)] ($g_l(0)$ is finite; $g_l(\infty)=0$), Jackson [19, p.125, (3.141)] ($g_m(0)$ is finite; $g_m(\infty)=0$), and Birkhoff–Rota [3, p.286, Theorem 12] are essentially the same. No wonder Jackson [19, p.120, l.1–l.9] and Jackson [19, p.125, l.−11–l.−3] have similar geometric interpretations for for region I: $\{\mathbf{x}|z<z'\}$ and region II: $\{\mathbf{x}|z>z'\}$. Jackson should have quoted Birkhoff–Rota [3, p.286, Theorem 12] whenever necessary instead of repeatng its proof many times.

6.3 The completeness theorem for self-adjoint problems on finite intervals

Theorem 6.2. Coddington–Levinson [7, p.197, Theorem 4.1]

Theorem 6.3. Coddington–Levinson [7, p.199, Theorem 4.2]

Remark. The difficult part that requires a proof is the following statement:
$u(t,l)$ is of class C^n in t (Coddington–Levinson [7, p.191, l.16]).

Proof. $\frac{\partial^n}{\partial^n t}u(t,l)=\int_a^t\frac{\partial^n K(t,\tau,l)}{\partial^n t}f(\tau)\,d\tau+\frac{\partial^{n-1}K(t,t^-,l)}{\partial^{n-1}t}f(t)$ (by the Leibniz integral rule)
$=\int_a^t\frac{\partial^n K(t,\tau,l)}{\partial^n t}f(\tau)\,d\tau+\frac{1}{p_0(t)}f(t)$ (by Coddington–Levinson [7, p.190, (2.4)]). □

6.4 The multiplicity and compatibility index of the characteristic numbers

For the homogeneous differential system of order n

$$\begin{cases}L(y)=0\\ U_i(y)=0 \quad (i=1,\cdots,n),\end{cases}$$

we may take its fundamental set of solutions $y_i(x,\lambda)(i=1,\cdots,n)$ [Ince [18, p.218, l.24]] such that $y_i(x,\lambda)$ is continuous on $\{(x,\lambda)|x\in[a,b],\lambda\in\mathbb{C}\}$ and for each fixed x, $y_i(x,\lambda)$ is an entire function in λ [Coddington–Levinson [7, p.37, l.19]]. Then the roots of the characteristic equation $F(\lambda)=0$ are isolated [Ince [18, p.218, l.−10]; Coddington–Levinson [7, p.190, l.19]]. For each root λ_i of $F(\lambda)$, its multiplicity m_i is greater or equal to its index of compatibility k_i [Ince [18, p.219, l.7]]. Ince [18, p.219, Theorem I; p.222, Theorem III] show how uniformly small variations in the coefficients of a linear differential system affect the index of the system.

Theorem 6.4. (Cases that the characteristic numbers of a system are real)(Ince [18, §10.7; §10.71])

(a) All the characteristic numbers in (A) (Ince [18, p.238]) are real

(i) if g is of one sign throughout the open interval (a,b) or

(ii) if $k>0, l\geq 0, \alpha\alpha'\geq 0, \beta\beta'\geq 0$.

(b) All the characteristic numbers in (B) (Ince [18, p.240]) are real if

(i) at least two of the ratios $\frac{\alpha_1}{\beta_1},\frac{\alpha_2}{\beta_2},\frac{\alpha_3}{\beta_3},\frac{\alpha_4}{\beta_4}$ are unequal,

(ii) $\gamma_1,\gamma_2,{\gamma_1}',{\gamma_2}'$ in (C) (Ince [18, p.240, l.15–l.16]) satisfy (D) (Ince [18, p.240, l.18]), and

(iii) $\gamma_1'\gamma_2 \geq 0$ *and* $\gamma_1\gamma_2 \geq 0$.

Corollary. (Characteristic numbers for a system with periodic boundary conditions are real)(Ince [18, p.241, l.1–l.5])

Condsider the system:

$$\begin{cases} \frac{d}{dx}\{k\frac{dy}{dx}\} + (\lambda g - l)y = 0, \\ y(a) = y(b), \\ y'(a) = y'(b). \end{cases}$$

If $k > 0, l \geq 0$ *and* $k(a) = k(b)$*, then the characteristic numbers of the system are real.*

6.5 There are an infinite number of characteristic numbers for a system with periodic boundary conditions

Theorem 6.5. Coddington–Levinson [7, p.214, Theorem 3.1]

Remark 1. The goal of Coddington–Levinson [7, p.212, Theorem 2.1] and that of Coddington–Levinson [7, p.214, Theorem 3.1] are the same, but their boundary conditions are different. The proof of the statement given in Coddington–Levinson [7, p.211, l.−2–l.−1] can be found in Ince [18, p.230, l.15–l.19]. Coddington–Levinson [7, p.213, l.9–l.16] should be replaced by Birkhoff–Rota [3, p.271, l.3–l.11] (Let $\gamma = \alpha, \gamma_1 = \pi - \delta, x_1 = c, \varepsilon = \delta$). The statement given in Coddington–Levinson [7, p.213, l.19] can be proved by Birkhoff–Rota [3, p.269, Figure 10.1].

Remark 2. Coddington–Levinson [7, p.215, (3.15)]

Proof. Since $f(\nu_0) \geq 2$, $f(\mu_0) \leq -2$, and f is continuous, $f([\nu_0, \mu_0])$ is connected. There exists a $\lambda \in [\nu_0, \mu_0)$ such that $f(\lambda_0) = 2$. □

Remark 3. If $\lambda_{2i+1} < \lambda_{2i+2}$ for some $i \geq 0$, then there is a unique eigenfunction φ_{2i+1} at $\lambda = \lambda_{2i+1}$ and a unique eigenfunction φ_{2i+2} at $\lambda = \lambda_{2i+2}$. If, however, $\lambda_{2i+1} = \lambda_{2i+2}$, then there are two independent eigenfunctions $\varphi_{2i+1}, \varphi_{2i+2}$ at $\lambda = \lambda_{2i+1} = \lambda_{2i+2}$ (Coddington–Levinson [7, p.214, l.4–l.7]).

Proof. $\lambda_{2i+1} \leq \mu_{2i+1} \leq \lambda_{2i+2}$ (Coddington–Levinson [7, p.215, l.−15]).
$\frac{df}{d\lambda}(\lambda_{2i+1}) \geq 0$ and $\frac{df}{d\lambda}(\lambda_{2i+2}) \leq 0$ (Coddington–Levinson [7, p.215, l.11]).

Case 1. ($\lambda_{2i+1} = \lambda_{2i+2}$)
Then $\frac{df}{d\lambda}(\mu_{2i+1}) = 0$.
By Coddington–Levinson [7, p.215, l.14–l.15], there are two independent eigenfunctions $\varphi_{2i+1}, \varphi_{2i+2}$ at $\lambda = \lambda_{2i+1} = \lambda_{2i+2}$.

Case 2. (Neither λ_{2i+1} nor λ_{2i+2} is equal to μ_{2i+1})
Then the result follows from Coddington–Levinson [7, p.215, l.9].

Case 3. ($\mu_{2i+1} = \lambda_{2i+1} < \lambda_{2i+2}$ or $\lambda_{2i+1} < \lambda_{2i+2} = \mu_{2i+1}$)
If $\frac{df}{d\lambda}(\mu_{2i+1}) = 0$, then $\mu_{2i+1} = \mu_{2i+2}$.
Then $\lambda_{2i+2} < \mu_{2i+2}$ (Coddington–Levinson [7, p.215, (3.15)])
$= \mu_{2i+1} \leq \lambda_{2i+2}$, a contradiction.
Hence $\frac{df}{d\lambda}(\mu_{2i+1}) \neq 0$ and the result follows from Coddington–Levinson [7, p.215, l.13].

□

Remark 4. If $f(\lambda^*)=2$ or -2 for some $\lambda^*\neq\mu_i$, then such a λ^* is a simple eigenvalue for (3.1) or (3.2) and for such a λ^*,
$\frac{df}{d\lambda}<0(\lambda^*<\mu_0);(-1)^i\frac{df}{d\lambda}>0(\mu_i<\lambda^*<\mu-i+1)(i=0,1,\ldots)$ (Coddington–Levinson [7, p.215, l.9–l.12]).

Proof. $\psi(1,\mu_i)\neq 0$ (Coddington–Levinson [7, p.217, l.9]).
$\frac{df}{d\lambda}(\lambda^*)$ has the same sign as $-\psi(1,\lambda^*)$ (Coddington–Levinson [7, p.217, l.11]).
λ^* is a simple eigenvalue (Coddington–Levinson [7, p.217, l.7–l.8]).
Since $\psi(0,\mu_i)=0,\psi'(0,\mu_i)=1,\psi(1,\mu_i)=0$, and $\psi(t,\mu_i)$ has i zeros in $(0,1)$ (Coddington–Levinson [7, p.216, l.3]),
we have $\psi(1,\lambda^*)<0$ if $\mu_i<\lambda^*<\mu_{i+1}$, where i is even.
This is because the zeros of $\psi(t,\lambda)$ move leftward (Coddington–Levinson [7, p.212, l.−12]) and the figure shape of $\psi(t,\lambda)$ will be preserved as λ increases slightly. □

Remark 5. $\psi(t,\mu_i)$ are the eigenfunctions (Coddington–Levinson [7, p.216, l.2]).

Proof. Let χ be the eigenfunction corresponding to μ_i.
($\chi_i(1,\mu_i)=0$ and $\psi_i(1,\mu_i)=0)\Rightarrow(\chi(t,\mu_i)$ and $\psi(t,\mu_i)$ are linearly dependent).
Otherwise, the general solution x of Coddington–Levinson [7, p.213, (3.1)] would satisfy $x(1)=0$, a contradiction. □

Remark 6. If ν_0 is the least eigenvalue of (2.1) with $x'(0)=x'(1)=0$, then $\varphi(t,\nu_0)$ is the eigenfunction and it has no zeros in $[0,1]$. Thus $\nu_0<\mu_0$ and $\varphi(1,\nu_0)>0$. (Coddington–Levinson [7, p.216, l.7–l.9])

Proof. $\varphi(t,\nu_0)$ has no zeros in $(0,1)$ (Coddington–Levinson [7, p.212, Theorem 2.1]).
$\varphi(0,\nu_0)=1$ (Coddington–Levinson [7, p.214, (3.4)]). If $\varphi(1,\nu_0)=0$, then φ and ψ are linearly dependent, a contradiction. Hence $\varphi(1,\nu_0)>0$. If $\nu_0>\mu_0$, then, by Theorem 5.1, $\varphi(t,\nu_0)$ has a zero in $(0,1)$, a contradiction. $\nu_0\neq\mu_0$ (Coddington–Levinson [7, p.214, (3.4)]). □

Remark 7. Because $\varphi(\tau)$ and $\psi(\tau)$ are independent, the bracket is identically zero if and only if all the coefficients vanish, which together with (3.5) is the condition (3.7) if $f=2$ (Coddington–Levinson [7, p.217, l.4–l.6]).

Proof. If there exists $(c,d)\neq(0,0)$ such that $(c\varphi(\tau)+d\psi(\tau))^2\equiv 0$, then $c\varphi(\tau)+d\psi(\tau)\equiv 0$, a contradiction.
$\varphi'(1)=0$, $\psi(1)=0$, and $\varphi(1)=\psi'(1)$ (Coddington–Levinson [7, p.216, the bracket in (3.17)]).
$\varphi(1)\psi'(1)=1$ (Coddington–Levinson [7, p.214, (3.5)]).
$\varphi(1)=\psi'(1)=1$ because $f(\lambda)=2$. □

Remark 8. The idea of proof method is given in Ince [18, p.143, l.1–l.3].

Remark 9. Let n_i be the number of roots of $F(\lambda)=0$ in (μ_{2i-1},μ_{2i+1}), where $i=1,2,\ldots$. Then n_i is even and greater than or equal to 2 (Ince [18, p.257, l.−12–l.−11]).
Note that the zeros of an entire function are isolated.

Remark 10. The inequalities given in Ince [18, p.244, l.18] follow from Theorem 5.1.

Remark 11. $\frac{\partial u}{\partial \lambda} = \frac{1}{K(a)} \int_a^x \frac{\partial G(t,\lambda)}{\partial \lambda} \{y_1(t,\lambda)y_2(x,\lambda) - y_2(t,\lambda)y_1(x,\lambda)\}$ (Ince [18, p.245, l.10]).

Proof. $K(t)(y_1(t,\lambda)y_2{}'(t,\lambda) - y_1{}'(t,\lambda)y_1(t,\lambda)|_a^x = 0$.
Therefore, $K(a)^{-1} = \Delta(y_1,y_2)^{-1}K(x)^{-1}$. □

Remark 12. $\frac{\partial}{\partial \lambda}(\frac{\partial u}{\partial x}) = \frac{1}{K(a)} \int_a^x \frac{\partial G(t,\lambda)}{\partial \lambda} \{y_1(t,\lambda)y_2{}'(x,\lambda) - y_2(t,\lambda)y_1{}'(x,\lambda)\}$ (Ince [18, p.245, l.12]).

Proof. The formula follows from the Leibniz integral rule. □

Remark 13. Note that the λ in the formula given in Ince [18, p.246, l.23] is a double root of $F(\lambda) = 0$.

Remark 14. $F(\lambda)$ preserves a constant negative sign in the neighborhood of a double characteristic number $\lambda = \lambda_0$ (Ince [18, p.246, l.−19–l.−18]).

Proof. Consider the Taylor series of $F(\lambda)$ at $\lambda = \lambda_0$. □

Remark 15. $y_2(b,\lambda)$ does not change sign at any interior point of (μ_{2m-1},μ_{2m}) (Ince [18, p.246, l.−9–l.−8]).

Proof. If $y_2(b,\lambda) = 0$, then $\lambda = \mu_i$, a contradiction. □

Remark 16. $F(\lambda) > 0$ in the interval $\Lambda_1 < \lambda < \nu_0$ (Ince [18, p.247, l.1]).

Proof. The statement follows from Coddington–Levinson [7, p.215, l.6 & l.11]. □

6.5.1 When the coefficients of the system's equation are periodic

Theorem 6.6. (Ince [18, p.248, l.6–l.7])
Characteristic number λ_i is ν_i if i is even, and is μ_{i+1} if i is odd.

Remark 1. The *existence* of $\{\lambda_i\}$ is proved in Ince [18, p.247, l.17–l.20]. Ince [18, pp.247–248, §10.81] tries to *identify* λ_i.

Remark 2. $\lambda = \nu_{2m}$ (Ince [18, p.248, l.1]) because $y_1(x,\lambda)$ is even (Ince [18, p.247, l.−13]).

Theorem 6.7. (Coddington–Levinson [7, p.219, l.−5–l.−4])
The stable regions are
$(\bigcup_{i=0}^{\infty}\{(a,b)|a_{2i}(b) < a < \tilde{a}_{2i+1}(b), -\infty < b < \infty\}) \cup (\bigcup_{i=1}^{\infty}\{(a,b)|\tilde{a}_{2i+2}(b) < a < a_{2i+1}(b), -\infty < b < \infty\})$.

Remark 1. The two equalities given in Coddington–Levinson [7, p.218, l.−14–l.−13] follow from Coddington–Levinson [7, p.79, (5.6)].

Remark 2. The statement given in Coddington–Levinson [7, p.219, l.1–l.3] follows from Coddington–Levinson [7, p.78, (5.3)].

Remark 3. Coddington–Levinson [7, p.219, (4.6)] corresponds to a solution which satisfies $x(0) = -x(1)$ and $x'(0) = -x'(1)$ and thus has period 2 (Coddington–Levinson [7, p.219, l.17–l.19]).

Proof. $\sigma = -1 = e^{i\pi}$. The result follows from Coddington–Levinson [7, p.79, (5.6)]. □

Remark 4. That the μ_i are continuous functions of b for each i follows from the fact that $\partial\psi/\partial a)(1,a.b) \neq 0$, where $\psi(1,a,b) = 0$ (Coddington–Levinson [7, p.219, l.-14–l.-13]).

Proof. Use Widder [42, p.44, Theorem 14]. □

Remark 5. The $\varphi(1,a,b)$ given in Coddington–Levinson [7, p.219, l.-11] does not vanish.

Proof. If $\varphi(1,a,b) = 0$, then $\varphi(t,a,b)$ and $\psi(t,a,b)$ are linearly dependent, a contradiction. □

7 Self-adjoint boundary-value problems for second-order singular equations

7.1 Introduction

Remark 1. As $b \to \infty$, $\rho_b(s) \to 2s/\pi$ [Coddington–Levinson [7, p.223, l.7]].

Proof. When s increases π/b, $\rho_b(s)$ increases $2/b$. Therefore, the slope is $2/\pi$. □

Remark 2. Coddington–Levinson [7, p.223, (1.6)] follows from Rudin [29, p.139, Theorem 7.16].

Remark 3. $\int_\mu^\infty s^{-2} d\rho_b(s) \leq \frac{4}{\pi}\int_\mu^\infty s^{-2} ds$ [Coddington–Levinson [7, p.223, (1.7)]].

Proof 1. $-\int_\mu^\infty \rho_b(s) d(s^{-2}) = -\int_\mu^\infty \frac{2}{b}[\frac{s}{\pi/b}](-2s^{-3}) ds \leq \frac{4}{\pi}\int_\mu^\infty s^{-2} ds.$
The result follows from Rudin [29, p.122, Theorem 6.30]. □

Proof 2. It suffices to prove $\int_{(\frac{(k-1)\pi}{b})^+}^{(\frac{k\pi}{b})^+} s^{-2} d\rho_b(s) \leq \frac{4}{\pi}\int_{\frac{(k-1)\pi}{b}}^{\frac{k\pi}{b}} s^{-2} ds.$
The left-hand side of this inequality is $(\frac{k\pi}{b})^{-2}\frac{2}{b}$. □

Remark 4. (1.4) can be proved for any $f \in \mathfrak{L}^2(0,\infty)$ [Coddington–Levinson [7, p.223, l.-2–l.-1]].

Proof. For every $s > 0$, let $g(s) = \int_0^\infty \sin st\, f(t) dt$.
By Rudin [31, p.71, Theorem 3.14] and Spivak [35, vol.1, p.43, (3)], there exists a sequence of functions f_n restricted as indicated in Coddington–Levinson [7, p.222, l.17; p.223, l.12–l.13] such that $||f_n - f|| \to 0$.
$\int_0^\infty |f_m(t) - f_n(t)|^2 dt = \frac{2}{\pi}\int_0^\infty |g_m(s) - g_n(s)|^2 ds.$
Since $\mathfrak{L}^2(0,\infty)$ is complete, there exists a $g \in \mathfrak{L}^2(0,\infty)$ such that $||g_n - g|| \to 0$.
Coddington–Levinson [7, p.223, (1.4)] is valid if we replace f with f_n and g with g_n. □

Remark 5. Coddington–Levinson [7, p.224, (1.9)].

Proof. $[fg] = \sum_{m=1}^{2} \sum_{\substack{j+k=m-1 \\ j\geq 0, k\geq 0}} (-1)^j f^{(k)} (a_{n-m}\bar{g})^{(j)}$ [Coddington–Levinson [7, p.86, (3.12)]]. □

7.2 The limit-point and limit-circle cases

Theorem 7.1. Coddington–Levinson [7, pp.228–229, Theorem 2.2 & Theorem 2.3]

Theorem 7.2. Coddington–Levinson [7, pp.229–230, Theorem 2.4]

Remark 1. Coddington–Levinson [7, p.225, (2.1)] follows from Coddington–Levinson [7, p.87, (6.15)].

Remark 2. The center of C_b is $\tilde{m}_b = \dfrac{A\bar{D} - B\bar{C}}{\bar{C}D - C\bar{D}}$ and the radius is $r_b = \dfrac{|AD - BC|}{|\bar{C}D - C\bar{D}|}$.

Proof. $|m - \dfrac{A\bar{D} - B\bar{C}}{\bar{C}D - C\bar{D}}|^2 = \dfrac{|AD - BC|^2}{|\bar{C}D - C\bar{D}|^2}$.
Namely, $[(\bar{C}D - C\bar{D})m - (A\bar{D} - B\bar{C})][(C\bar{D} - \bar{C}D)\bar{m} - (\bar{A}D - \bar{B}C)] = (AD - BC)(\bar{A}\bar{D} - \bar{B}\bar{C})$.
Identify this equality with $(C\bar{D} - \bar{C}D)\times$ (the equality given in Coddington–Levinson [7, p.227, l.2]). □

Remark 3. m is on the limit circle if and only if $[\chi\chi](\infty) = 0$ [Coddington–Levinson [7, p.228, l.-11–l.-10]].

Proof. By Coddington–Levinson [7, p.224, (1.9)], $(l - \bar{l}) \int_0^\infty |\chi|^2 dt = [\chi\chi](\infty) - [\chi\chi](0)$.
The result follows from Coddington–Levinson [7, p.227, (2.11)] and the equality given in Coddington–Levinson [7, p.227, l.-4]. □

Remark 4.

$$G(t,\tau,l) = \begin{cases} \psi(t,l)\{\varphi(\tau,l) + m(l,b,\beta)\psi(\tau,l)\} & (t \leq \tau) \\ \psi(\tau,l)\{\varphi(t,l) + m(l,b,\beta)\psi(t,l)\} & (t > \tau) \end{cases}$$ [Coddington–Levinson [7, p.229, l.13]].

Proof. Note that the coefficient of x'' of Lx given in Coddington–Levinson [7, p.231, (3.1)] is $-p$.
By Coddington–Levinson [7, p.226, l.11], $[\varphi\bar{\psi}] = 1$.
Since $\psi(t,l)$ satisfies the boundary condition given in Coddington–Levinson [7, p.229, l.10] and $\varphi + m(l,b,\beta)\psi(t,l)$ satisfies the boundary condition given in Coddington–Levinson [7, p.229, l.11], $\psi(t,l)$ and $\varphi + m(l,b,\beta)\psi(t,l)$ satisfy the conditions given in Birkhoff–Rota [3, p.285, l.14–l.16]. The result follows from Birkhoff–Rota [3, p.286, (67)]. □

Remark 5. The points $m = m(l,b,\beta)$ on C_b are uniformly bounded as $b \to \infty$ [Coddington–Levinson [7, p.229, l.-14–l.-13]].

Proof. Since Λ is compact and $[\chi\chi](1) \neq 0$ [Coddington–Levinson [7, p.227, l.-7]],

$$M_1 = \max_{l \in \Lambda} |\tilde{m}_1| < \infty$$

and

$$M_2 = \max_{l \in \Lambda} |r_1| < \infty.$$

Then $|m_b| \leq M_1 + M_2 \quad (b > 1, l \in \Lambda)$. □

Remark 6. The functions m_b are equicontinuous on Λ [Coddington–Levinson [7, p.229, l.-12–l.-11]].

Proof.

$$\begin{aligned} m_b(l) - m_b(l') &= \frac{1}{2\pi i} \int_C \left(\frac{1}{\zeta - l} - \frac{1}{\zeta - l'}\right) m_b(\zeta) d\zeta \\ &= \frac{l - l'}{2\pi i} \int_C \frac{m_b(\zeta)}{(\zeta - l)(\zeta - l')} d\zeta, \end{aligned}$$

where C is the boundary of a disk $D(u : r)$ and l, $l' \in D(u; \frac{r}{2})$. Then
$|m_b(l) - m_b(l')| \leq \frac{4(M_1+M_2)|l-l'|}{r}$. □

Remark 7. m_b converges uniformly to m_∞ [Coddington–Levinson [7, p.229, l.-11–l.-10]].

Proof. The statement follows from Hartman [17, p.4, Selection Theorem 2.3; Remark 2]. □

Remark 8. If m_∞ has zeros or poles on the real axis, they are simple [Coddington–Levinson [7, p.229, l.-7]].

Proof. The statement follows from Rudin [31, p.225, Theorem 10.18]. □

Remark 9. The poles have negative residue [Coddington–Levinson [7, p.229, l.-6]].

Proof. Let c be a real pole and l a real number. Then

$$Res(m_\infty; c) = \lim_{l \to c} (l - c) m_b(l)$$

is real [Coddington–Levinson [7, p.226, (2.5); l.11]].

$$\begin{aligned} (l \to c, \Im l > 0) &\Rightarrow \Im\left(\frac{Res(m_\infty; c)}{l - c}\right) > 0 \quad \text{[Coddington–Levinson [7, p.229, l.}-8\text{]]} \\ &\Rightarrow -Res(m_\infty; c) \Im l > 0. \end{aligned}$$

□

7.3 The completeness and expansion theorems in the limt-point case at infinity

Theorem 7.3. Coddington–Levinson [7, p.233, l.−2–l.−1].

Remark 1. This is a *self-adjoint* boundary value problem on $0 \leq t \leq b$ [Coddington–Levinson [7, p.231, l.22]].

Proof. Read Ince [18, p.217, l.1]. □

Remark 2. There exists a sequence of $\{\lambda_{bn}\}, n = 1, 2, \ldots$, of real eigenvalues and a corresponding complete orthonormal set of $\{\theta_{bn}\}$ of eigenfunctions [Coddington–Levinson [7, p.231, l.−10–l.−8]].

Proof. Read Coddington–Levinson [7, p.189, Theorem 2.1; p.197, l.8; p.199, Theorem 4.2]. □

Remark. There can be *exactly one* eigenfunction corresponding to the eigenvalue λ_{bn}. If there were two, they would have to be linearly dependent because they would satisfy the same differential equation and the first boundary condition of Coddington–Levinson [7, p.231, (3.1)]. Then $\{\theta_{bn}\}$ could not be orthonormal.

Remark 3. No solution of $Lx = lx$ independent of ψ can satisfy the first boundary condition of (3.1) [Coddington–Levinson [7, p.231, l.−10–l.−9]].

Proof. If there were to exist a solution of $Lx = lx$ independent of ψ can satisfy the first boundary condition of (3.1), then φ would also satisfy the first boundary condition of (3.1). However, by Coddington–Levinson [7, p.226, (2.2)], $\sin\alpha\varphi(0) - \cos\alpha p(0)\varphi'(0) = 1$. □

Remark 4. Coddington–Levinson [7, p.231, (3.2)].

Proof. If f is given as in Coddington–Levinson [7, p.235, l.12–l.13], we use Coddington–Levinson [7, p.198, Corollary].
If f is given as in Coddington–Levinson [7, p.232, Theorem 3.1(ii)], we use the equality given in Coddington–Levinson [7, p.199, l.−5]. □

Remark 5. By Coddington–Levinson [7, p.232, (3.8)] and Rudin [31, p.70, Theorem 3.12], the "=" sign given in Coddington–Levinson [7, p.233, (3.12)] can be interpreted as "being equal almost everywhere".

Remark 6. The Helly selection theorem [Coddington–Levinson [7, p.233, l.−9]].

The construction of h.

1 Let $\mathbb{Q}$ be the set of rationals. There is a subsequence $\{g_r | r \in \mathbb{N}\}$ of $\{h_n\}$ such that

$$h(x) = \lim_{r\to\infty} g_r(x)$$

exists for every $x \in \mathbb{Q}$.

2 $\forall x \notin \mathbb{Q}$, let

$$h(x) = \sup_{\substack{t\in\mathbb{Q}\\ t<x}} h(t).$$

Then $h(x)$ is nondecreasing.

3 Let $C(h) = \{x \in (-\infty,\infty) | h \text{ is continuous at } x\}$. Then

$$\forall x \in C(h), \lim_{r\to\infty}(x) = h(x).$$

4 Let F be the complement set of $C(h)$. Then F is countable. There is a subsequence $\{k_m | m \in \mathbb{N}\}$ of $\{g_r\}$ such that $\lim_{m\to\infty} k_m(x)$ exists for every $x \in F$. Redefine $h(x) = \lim_{m\to\infty} k_m(x) \quad (x \in F)$.

□

Remark 7. By Coddington–Levinson [7, p.237, (3.23)], $f \leftrightarrow g$ is a *norm preserving* mapping of $\mathfrak{L}^2(0,\infty)$ onto $\mathfrak{L}^2(\rho)$ [Coddington–Levinson [7, p.233, l.−2–l.−1]].

Remark 8. Integration theorem [Coddington–Levinson [7, p.234, l.5]].

Proof. Let $\mu_n(x,y] = h_n(y) - h_n(x)$. Then use Royden [28, p.232, Proposition 18]. □

Remark 9. Both χ_b and ψ_{bn} satisfy the same boundary condition at b [Coddington–Levinson [7, p.234, l.−9]].

Proof. χ_b satisfies Coddington–Levinson [7, p.226, (2.4)].
ψ_{bn} satisfies the first boundary condition of Coddington–Levinson [7, p.231, (3.1)] [Coddington–Levinson [7, p.231, l.−11–l.−10]]. □

Remark 10. $[\chi_b\psi_{bn}](0) = 1$.

Proof. By Coddington–Levinson [7, p.231, l.−12–l.−11],
$\chi_b(0) = \sin\alpha + m_b\cos\alpha$ and $p(0)\chi_b'(0) = -\cos\alpha + m_b\sin\alpha$.
$[\chi_b\psi_{bn}](0) = p(0)(\chi_b(0)\psi_{bn}'(0) - \chi_b'(0)\psi_{bn}(0)) = \sin\alpha(\sin\alpha + m_b\cos\alpha) - (-\cos\alpha + m_b\sin\alpha)\cos\alpha$.
□

Remark 11. There exists a sequence of functions $f_n \in \mathfrak{L}^2(0,\infty)$ possessing continuous second derivatives and vanishing near $t = 0$ and for all large t such that

$$\lim_{n\to\infty}\int_0^\infty |f_n - f|^2 dt = 0$$

[Coddington–Levinson [7, p.236, l.6–l.9]].

Proof. Use Rudin [31, p.16, Theorem 1.17] and Spivak [35, vol. 1, p.43, (3)]. □

Remark 12. g is the continuous function given by $g(\lambda) = \int_0^\infty f(t)\psi(t,\lambda)dt$ [Coddington–Levinson [7, p.236, l.17–l.19]].

Proof. g is well-defined by Rudin [31, p.16, Theorem 1.17; p.27, Theorem 1.34].
Assume $f \equiv 0$ on $(-\infty, \infty) \setminus [-\mu, \mu]$.
Let

$$M = \max_{\substack{t \in [-\mu,\mu] \\ \lambda \in \Lambda \text{ compact}}} |\psi(t, \lambda)|.$$

$$\begin{aligned} |g(\lambda_m) - g(\lambda)| &\leq |g(\lambda_m) - g_n(\lambda_m)| + |g_n(\lambda_m) - g_n(\lambda)| + |g_n(\lambda) - g(\lambda)| \\ &\leq M\|f_n - f\|_1 + |g_n(\lambda_m) - g_n(\lambda)| + M\|f_n - f\|_1. \end{aligned}$$

□

Remark 13. For any fixed l, $\Im l > 0$, there exists a constant k such that $\int_{-\mu}^{\mu} \frac{d\rho_b(\lambda)}{|\lambda - l|^2} \leq k$ for $b > 1$ and all $\mu \geq 0$ [Coddington–Levinson [7, p.238, l.7–l.10]].

Proof. The result follows from the statement given in Coddington–Levinson [7, p.228, l.5–l.6]. □

Remark 14. (3.25) must tend to $\int_{-\infty}^{\infty} (\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l_0|^2}) d\rho(\lambda)$ [Coddington–Levinson [7, p.238, l.−9–l.−8]].

Proof. There exists a constant $c > 0$ such that $\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l_0|^2} \leq \frac{c}{|\lambda|^3}$. This is because on the left side of the inequality, the leading term of the numerator is λ, while the leading term of the denominator is λ^4.
$\int_{\mu}^{\infty} (\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l_0|^2}) d\rho_b(\lambda) \leq \int_{\mu}^{\infty} \frac{c}{|\lambda|^3} d\rho_b(\lambda) \leq \frac{ck}{\mu}$.
By Royden [28, p.231, Proposition 17], $\int_{\mu}^{\infty} \frac{c}{|\lambda|^3} d\rho(\lambda) \leq \frac{ck}{\mu}$.
$\int_{-\mu}^{\mu} (\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l_0|^2}) d\rho_b(\lambda) \leq 2k$ [Coddington–Levinson [7, p.238, l.9]].
$\int_{-\mu}^{\mu} (\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l_0|^2}) d\rho_b(\lambda) \to \int_{-\mu}^{\mu} (\frac{1}{|\lambda - l|^2} - \frac{1}{|\lambda - l_0|^2}) d\rho(\lambda)$ as $b \to \infty$ [Royden [28, p.231, Proposition 17; p.232, Proposition 18]]. □

Remark 15. Coddington–Levinson [7, p.239, (3.27)].

Proof.

$$\Im(\frac{1}{\lambda - l}) = \frac{1}{2i}(\frac{1}{\lambda - l} - \frac{1}{\lambda - \bar{l}}) = \frac{1}{2i} \frac{l - \bar{l}}{|\lambda - l|^2} = \frac{\Im l}{|\lambda - l|^2}.$$

□

Remark 16.

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} [\arctan(\frac{\lambda - \sigma}{\varepsilon}) - \arctan(\frac{\lambda - \sigma}{\varepsilon})] d\rho(\sigma) = \pi(\rho(\mu) - \rho(\mu)).$$

Proof. Assume $\mu < \lambda$.

$$\text{The quantity in the square brackets} = \begin{cases} \frac{\pi}{2} - \frac{\pi}{2} & \text{if } \sigma < \mu \\ \frac{\pi}{2} - (-\frac{\pi}{2}) & \text{if } \mu \leq \sigma \leq \lambda \\ -\frac{\pi}{2} - \frac{\pi}{2} & \text{if } \lambda < \sigma. \end{cases}$$

□

Remark 17. $f_\triangle \in \mathfrak{L}^2(0,\infty)$ [Coddington–Levinson [7, p.239, l.-15]].

Proof. $\int_0^\infty f_\triangle \bar{P} dt = \int_\triangle g(\lambda)\bar{Q}(\lambda)d\rho(\lambda)$ [Coddington–Levinson [7, p.237, (3.23)]].
$(\int_0^a |f_\triangle(t)|^2 dt)^2 \le \int_\triangle |g(\lambda)|^2 d\rho(\lambda) \int_{-\infty}^\infty |Q(\lambda)|^2 d\rho(\lambda)$.
$\int_{-\infty}^\infty |Q(\lambda)|^2 d\rho(\lambda) = \int_{-\infty}^\infty |P(t)|^2 dt = \int_0^a |f_\triangle(t)|^2 dt$. □

Remark 18. If $\tilde{\psi} \in \mathfrak{L}^2(0,\infty)$, the jump of ρ_b at $\tilde{\lambda}$ does not tend to zero as $b \to \infty$ [Coddington–Levinson [7, p.258, l.-12]].

Proof. Use Coddington–Levinson [7, p.231, (3.2)]. □

Remark 19. The f given in Coddington–Levinson [7, p.240, l.2] refers to the f defined by Coddington–Levinson [7, p.239, l.-13] instead of the f given in Coddington–Levinson [7, p.232, Theorem 3.1(ii)].

Remark 20. Coddington–Levinson [7, p.240, (3.28)].

Proof. $\|f_\triangle - f\| \to 0$ as $\triangle \to (-\infty,\infty)$ [Coddington–Levinson [7, p.239, Lemma 3.1]].
$\|\tilde{f}_\triangle - f\| \to 0$ as $\triangle \to (-\infty,\infty)$ [Coddington–Levinson [7, p.232, Theorem 3.1(iii)]]. □

Remark 21. We may use Coddington–Levinson [7, p.87, (6.15); l.226, l.11; p.240, l.-9–l.-8] to prove Coddington–Levinson [7, p.240, (3.31)].

Remark 22. $\|H_\triangle - H\|_2 \to 0$ as $\triangle \to (-\infty,\infty)$ [Coddington–Levinson [7, p.240, l.-3–l.-1]].

Proof. Use Coddington–Levinson [7, p.239, Lemma 3.1]. □

Remark 23. $H(t,l) = c\psi(t,l)$ [Coddington–Levinson [7, p.241, l.2]].

Proof. Apply the Schwarz inequality to Coddington–Levinson [7, p.240, (3.31)]. □

Remark 24. Coddington–Levinson [7, p.241, (3.32)].

Proof.

$$\int_\triangle \frac{r(\lambda)}{\lambda - l}\Gamma_s(\lambda)d\rho(\lambda) = \int_\triangle \frac{r(\lambda)}{\lambda - l}\int_0^s \psi(t,\lambda)dtd\rho(\lambda) = \int_0^s H_\triangle(t,l)dt.$$

$$\int_0^s |H_\triangle(t,l)|dt \le \|H_\triangle\|_2 \|\chi_{[0,s]}\|_2.$$

□

7.4 The limit-circle case at infinity

Theorem 7.4. Coddington–Levinson [7, pp.242–243, Theorem 4.1].

Theorem 7.5. Coddington–Levinson [7, p.245, l.-2–l.-1].

Theorem 7.6. Coddington–Levinson [7, p.246, Theorem 4.3].

Remark 1. In case $\Im l = 0$, the circle $C_b(l)$ becomes a straight line [Coddington–Levinson [7, p.242, l.-18–l.-17]].

Proof. By [Coddington–Levinson [7, p.227, (2.11)]], $\Im l = 0 \Rightarrow \Im m = 0$. □

Remark 2. Coddington–Levinson [7, p.243, (4.5)].

Proof. $\int_0^{b_j}[\chi_j(t,l_0)L\chi(t,l) - \chi_j(t,l)L\chi(t,l_0)]dt$
$= [\chi_j(t,l)\bar{\chi}_j(t,l_0)](b_j) - [\chi_j(t,l)\bar{\chi}_j(t,l_0)](0)$ [Coddington–Levinson [7, p.224, (1.9)]]
$= -[\chi_j(t,l)\bar{\chi}_j(t,l_0)](0)$ [Coddington–Levinson [7, p.243, l.13–l.14]].
The result follows from [Coddington–Levinson [7, p.226, (2.2)]]. □

Remark 3. φ and ψ have norms in $\mathfrak{L}^2(0,\infty)$ which are uniformly bounded in Λ [Coddington–Levinson [7, p.243, l.-7–l.-6]].

Proof. $\|\varphi(\cdot,l)\|_2$ is continuous on Λ. □

Remark 4. The integrals in (4.7) are uniformly convergent in l over any finite part of the l plane [Coddington–Levinson [7, p.243, l.-5–l.-4]].

Proof. $\int_c^\infty |\varphi(t,l)[\varphi(t,l_0) + \hat{m}_\infty(l_0)\psi(t,l_0)]|dt \le \|\varphi(\cdot,l)\|_2(\int_0^\infty |\varphi(t,l_0) + \hat{m}_\infty(l_0)\psi(t,l_0)|^2 dt)^{1/2}$.
$\|\varphi(\cdot,l)\|_2$ is uniformly bounded on Λ and
$(\int_0^\infty |\varphi(t,l_0) + \hat{m}_\infty(l_0)\psi(t,l_0)|^2 dt)^{1/2} \to 0$ as $c \to \infty$ [Coddington–Levinson [7, p.225, l.-5]]. □

Remark 5. $\Im \hat{m}_\infty(l)/\Im l > 0$ for $\Im l \ne 0$ [Coddington–Levinson [7, p.244, l.1]].

Proof. Let Λ be a compact subset on $\Im l > 0$. By Coddington–Levinson [7, p.227, (2.11)], $\Im m(l,1,\beta)/\Im l$ has a minimum > 0 on $\Lambda \times [-\pi,\pi]$. The result follows from Coddington–Levinson [7, p.228, l.5–l.6]. □

Remark 6. By Coddington–Levinson [7, p.244, l.1], $\hat{m}_\infty$ is real on the real axis [Coddington–Levinson [7, p.244, l.7–l.8]].

Remark 7. Let $\mu = \lambda_k^-$ and $\lambda = \lambda_k^+$. By González [15, p.683, Lemma 9.4], the right side of the equality given in Coddington–Levinson [7, p.232, (3.9)] equals to $-Res(\hat{m}_\infty;\lambda_k)$.

Remark 8.

$$\hat{G}(t,\tau,l_0) = \begin{cases} \psi(t,l_0)\hat{\chi}_\infty(\tau,l_0) & (t \le \tau) \\ \psi(\tau,l_0)\hat{\chi}_\infty(t,l_0) & (t > \tau) \end{cases}$$ [Coddington–Levinson [7, p.244, (4.10)]].

Proof. Note that the coefficient of x'' of Lx given in Coddington–Levinson [7, p.231, (3.1)] is $-p$.
By Coddington–Levinson [7, p.226, l.11], $[\varphi\bar{\psi}] = 1$.
Since $\psi(t, l_0)$ satisfies the boundary condition given in Coddington–Levinson [7, p.244, (iii)] and $\hat{\chi}_\infty(t, l_0)$ satisfies the boundary condition given in Coddington–Levinson [7, p.244, (iv)], $\psi(t, l_0)$ and $\hat{\chi}_\infty(t, l_0)$ satisfy the conditions given in Birkhoff–Rota [3, p.285, l.14–l.16]. The result follows from Birkhoff–Rota [3, p.286, (67)]. □

Remark 9. $[\psi\hat{\chi}_\infty](\infty) = (\bar{\tilde{m}}_\infty - \tilde{m}_\infty)[\psi\psi](\infty)$ [Coddington–Levinson [7, p.245, l.12]].

Proof. $[\psi\hat{\chi}_\infty](\infty) = [\psi\varphi](\infty) + \bar{\tilde{m}}_\infty[\psi\psi](\infty)$.
$[\psi\varphi](\infty) = -[\bar{\varphi}\bar{\psi}](\infty)$
$= \tilde{m}_\infty[\bar{\psi}\bar{\psi}](\infty)$ [Coddington–Levinson [7, p.245, (4.12)]]
$= -\bar{\tilde{m}}_\infty[\psi\psi](\infty)$. □

Remark 10. $u'(t) = \hat{\chi}'_\infty(t, l_0)\int_0^\infty \psi(\tau, l_0)f(\tau)d\tau + \psi'(t, l_0)\int_t^\infty \hat{\chi}_\infty(\tau, l_0)f(\tau)d\tau$.

Proof. By the Leibniz integral rule,
$\frac{d}{dt}\int_0^t \psi(\tau, l_0)\hat{\chi}_\infty(t, l_0)f(\tau)d\tau = \hat{\chi}'_\infty(t, l_0)\int_0^t \psi(\tau, l_0)f(\tau)d\tau + \hat{\chi}_\infty(t, l_0)\psi(t, l_0)f(t)$ and
$\frac{d}{dt}\int_t^\infty \psi(t, l_0)\hat{\chi}_\infty(\tau, l_0)f(\tau)d\tau = \psi'(t, l_0)\int_t^\infty \hat{\chi}_\infty(\tau, l_0)f(\tau)d\tau + \psi(t, l_0)(-\hat{\chi}_\infty(t, l_0)f(t))$. □

Remark 11. If the two functions are absolutely continuous on a bounded closed interval, then their product is also absolutely continuous. Therefore, u' is absolutely continuous on $[0, b]$ for all $b < \infty$ [Coddington–Levinson [7, p.245, l.18]].

Remark 12. By Coddington–Levinson [7, p.245, l.18], $[\psi\hat{\chi}_\infty](0) = -1$.

Remark 13. $\hat{\mathscr{G}}(l_0)f \in \mathfrak{L}^2(0, \infty)$ [Coddington–Levinson [7, p.245, l.−14]].

Proof. By Coddington–Levinson [7, p.244, (4.10)],
$\hat{\mathscr{G}}(l_0)f(t) = \hat{\chi}_\infty(t, l_0)\int_0^t \psi(\tau, l_0)f(\tau)d\tau + \psi(t, l_0)\int_t^\infty \hat{\chi}_\infty(\tau, l_0)f(\tau)d\tau$.
By the Schwarz inequality,
$\int_0^\infty |\psi(\tau, l_0)f(\tau)|d\tau \le \|\psi\|_2\|f\|_2$ and
$\int_0^\infty |\hat{\chi}_\infty(\tau, l_0)f(\tau)|d\tau \le \|\hat{\chi}_\infty\|_2\|f\|_2$. Hence
$|\hat{\mathscr{G}}(l_0)f(t)| \le |\hat{\chi}_\infty(t, l_0)| \le \|\psi\|_2\|f\|_2 + |\psi(t, l_0)|\|\hat{\chi}_\infty\|_2\|f\|_2$
$\in \mathfrak{L}^2(0, \infty)$ (by Minkowski inequality). □

Remark 14. By Coddington–Levinson [7, p.244, (4.10); p.245, l.17], u satisfies Coddington–Levinson [7, p.244, (iii)] [Coddington–Levinson [7, p.245, l.−12]].

Remark 15. u satisfies Coddington–Levinson [7, p.244, (iv)] [Coddington–Levinson [7, p.245, l.−11]].

Proof. $[u\hat{\chi}_\infty](\infty) = p(\infty)[u(\infty)\bar{\hat{\chi}}'_\infty(\infty) - u'(\infty)\bar{\hat{\chi}}_\infty(\infty)]$
$= [\hat{\chi}_\infty\hat{\chi}_\infty](\infty)\int_0^\infty \psi(\tau, l_0)f(\tau)d\tau$ [Coddington–Levinson [7, p.244, (4.10); p.245, l.17]]. □

Remark 16. $Lw - l_0w = 0$ [Coddington–Levinson [7, p.245, l.−8]].

Proof. $Lw = Lu - L\hat{\mathscr{G}}(l_0)f$ (by definition) $= f + l_0 u - L\hat{\mathscr{G}}(l_0)f$ [Coddington–Levinson [7, p.245, l.-10]] $= f + (l_0 w + l_0\hat{\mathscr{G}}(l_0)f)L\hat{\mathscr{G}}(l_0)f$ (by definition) $= f + l_0 w - f$ (by Coddington–Levinson [7, p.245, l.1], $(L - l_0)\hat{\mathscr{G}}(l_0)f = f$). ☐

Remark 17. $c_1 = c_2 = 0$ [Coddington–Levinson [7, p.245, l.-6]].

Proof. If $c_1 \neq 0$, then ψ satisfies Coddington–Levinson [7, p.244, (iv)]. This would contradict Coddington–Levinson [7, p.245, (4.13)].
$(c_2 \neq 0) \Rightarrow \hat{\chi}_\infty$ satisfies Coddington–Levinson [7, p.244, (iii)]
$\Rightarrow \varphi$ satisfies Coddington–Levinson [7, p.244, (iii)]. This would contradict Coddington–Levinson [7, p.226, (2.2)]. ☐

Remark 18. Since both u and v satisfy Coddington–Levinson [7, p.244, (iii)], $[uv](0) = 0$ [Coddington–Levinson [7, p.246, l.5]].

Remark 19. Coddington–Levinson [7, p.246, (4.16)].

Proof. $p(b)(u(b)\bar{v}'(b) - u'(b)\bar{v}(b))$
$= p(b)\{(\hat{\chi}_\infty(b,l_0)\int_0^b \psi(\tau,l_0)f(\tau)d\tau + \psi(b,l_0)\int_b^\infty \hat{\chi}_\infty(\tau,l_0)f(\tau)d\tau)$
$(\overline{\hat{\chi}'_\infty(b,l_0)}\int_0^b \overline{\psi(\tau,l_0)g(\tau)}d\tau + \overline{\psi'(b,l_0)}\int_b^\infty \overline{\hat{\chi}_\infty(\tau,l_0)g(\tau)}d\tau)$
$-(\hat{\chi}'_\infty(b,l_0)\int_0^b \psi(\tau,l_0)f(\tau)d\tau + \psi'(b,l_0)\int_b^\infty \hat{\chi}_\infty(\tau,l_0)f(\tau)d\tau)$
$(\overline{\hat{\chi}_\infty(b,l_0)}\int_0^b \overline{\psi(\tau,l_0)g(\tau)}d\tau + \overline{\psi(b,l_0)}\int_0^b)\}\int_b^\infty \overline{\hat{\chi}_\infty(\tau,l_0)g(\tau)}d\tau)\}$
$= p(b)\{[\hat{\chi}_\infty\hat{\chi}_\infty](b)(\int_0^b \psi(\tau,l_0)f(\tau)d\tau)(\int_0^b \overline{\psi(\tau,l_0)g(\tau)}d\tau)$
$+ [\psi\hat{\chi}_\infty](b)(\int_b^\infty \hat{\chi}_\infty(\tau,l_0)f(\tau)d\tau)(\int_0^b \overline{\psi(\tau,l_0)g(\tau)}d\tau)$
$+ [\hat{\chi}_\infty\psi](b)(\int_0^b \psi(\tau,l_0)f(\tau)d\tau)(\int_b^\infty \overline{\hat{\chi}_\infty(\tau,l_0)g(\tau)}d\tau)$
$+ [\psi\psi](b)(\int_b^\infty \hat{\chi}_\infty(\tau,l_0)f(\tau)d\tau)(\int_b^\infty \overline{\hat{\chi}_\infty(\tau,l_0)g(\tau)}d\tau)\}$.
The first term approaches to 0 as $b \to \infty$ because of Coddington–Levinson [7, p.245, (4.11)]. The last three terms approach to 0 as $b \to \infty$ because $\int_b^\infty \to 0$ as $b \to \infty$. ☐

Remark 20. By Coddington–Levinson [7, p.243, l.1], ψ_k are complete [Coddington–Levinson [7, p.246, l.15]].

7.5 Singular behavior at both ends of an interval

Theorem 7.7. Coddington–Levinson [7, p.251, Theorem 5.1].

Theorem 7.8. Coddington–Levinson [7, pp.251–252, Theorem 5.2].

Remark 1. Coddington–Levinson [7, p.247, l.-9, (ii)].

Proof 1.

$$\begin{vmatrix} \sum_{m\le i\le n} r_{\delta i1}\bar{r}_{\delta i1} & \sum_{m\le i\le n} r_{\delta i1}\bar{r}_{\delta i2} \\ \sum_{m\le i\le n} r_{\delta i2}\bar{r}_{\delta i1} & \sum_{m\le i\le n} r_{\delta i2}\bar{r}_{\delta i2} \end{vmatrix}$$

$$
\begin{aligned}
&= \sum_{m\le i\le n}\sum_{m\le j\le n} r_{\delta i1}r_{\delta j2}\bar r_{\delta i1}\bar r_{\delta j2} - \sum_{m\le i\le n}\sum_{m\le j\le n} r_{\delta i1}r_{\delta j2}\bar r_{\delta i2}\bar r_{\delta j1}\\
&= \sum_{m\le i<j\le n} r_{\delta i1}r_{\delta j2}(\bar r_{\delta i1}\bar r_{\delta j2}-\bar r_{\delta i2}\bar r_{\delta j1}) + \sum_{m\le j<i\le n} r_{\delta i1}r_{\delta j2}(\bar r_{\delta i1}\bar r_{\delta j2}-\bar r_{\delta i2}\bar r_{\delta j1})\\
&= \sum_{m\le i<j\le n} (r_{\delta i1}r_{\delta j2}-r_{\delta i2}r_{\delta j1})(\bar r_{\delta i1}\bar r_{\delta j2}-\bar r_{\delta i2}\bar r_{\delta j1}).
\end{aligned}
$$

□

Proof 2. Since the sum of positive semidefinite matrices is positive semidefinite, it suffices to observe

$$\begin{bmatrix} r_1\bar r_1 & r_1\bar r_2\\ r_2\bar r_1 & r_1\bar r_2\end{bmatrix} = \begin{bmatrix} r_1 & 0\\ r_2 & 0\end{bmatrix}\begin{bmatrix} \bar r_1 & \bar r_2\\ 0 & 0\end{bmatrix}.$$

□

Remark. The critical structure may fail to emerge more often because of lacking in skilful analysis. The second proof can be generalized to prove Coddington–Levinson [7, p.263, l.4, (ii)].

Remark 2. There is the usual nonuniqueness [Coddington–Levinson [7, p.248, l.7–l.8]].

Proof. By Coddington–Levinson [7, p.244, l.5–l.6], $\hat m_\infty$ can be any point on the limit circle. □

Remark 3. C_a is in C_{-1} for $a<-1$ [Coddington–Levinson [7, p.250, l.10–l.11]].

Proof. By [Coddington–Levinson [7, p.227, (2.9)]],
(m_a is in the interior of C_a) if and only if $\frac{[\chi_a\chi_a](a)}{[\psi\psi](a)}<0$.

$$\frac{[\chi_a\chi_a](a)}{[\psi\psi](a)} = \frac{\int_0^a|\chi_a|^2dt}{\int_0^a|\psi|^2dt} - \frac{\Im m_a}{(\Im l)\int_0^a|\psi|^2dt}.$$

Therefore, (m_a is in the interior of C_a) if and only if $\int_0^a|\chi_a|^2dt > \frac{\Im m_a}{\Im l}$. □

Remark 4. Helly selection theorem for functions with dominated total variation [Coddington–Levinson [7, p.250, l.−10–l.−5]].

Let H be a continuous nonnegative function on $(-\infty,\infty)$ and $\{h_n\}, n=1,2,\cdots$, be a sequence of functions on $(-\infty,\infty)$ such that $|h_n(\lambda)|\le H(\lambda)(n=1,2,\cdots;-\infty<\lambda<\infty)$ and $V(h_n;-\lambda,\lambda)\le H(\lambda)(n=1,2,\cdots;0\le\lambda<\infty)$. Then there exist a subsequence $\{h_{n_k}\}$ and a function h such that $\lim_{k\to\infty}h_{n_k}(\lambda)=h(\lambda)$. If we fix λ_0, then $V(h;-\lambda_0,t)\le 3H(t)(-\lambda_0\le t\le\lambda_0)$.

Proof. By Coddington–Levinson [7, pp.233–234, Selection Theorem], there exist a subsequence $V(h_{n_r};-\lambda,\lambda)$ and a nondecreaing function α such that $\lim_{r\to\infty}V(h_{n_r};-\lambda,t)=\alpha(t)\,(t\in[-\lambda,\lambda])$.
$V(h_{n_r};-\lambda,t)-h_{n_r}(t)$ is nondecreasing on $[-\lambda,\lambda]$.
$V(h_{n_r};-\lambda,t)-h_{n_r}(t)\le V(h_{n_r};-\lambda,t)+|h_{n_r}(t)|\le 2H(t)$.
By Coddington–Levinson [7, pp.233–234, Selection Theorem], there exist a subsequence $V(h_{n_{r_j}};-\lambda,\lambda)$ of $V(h_{n_{r_j}};-\lambda,\lambda)$ and a nondecreaing function β such that $\lim_{j\to\infty}V(h_{n_{r_j}};-\lambda,t)-h_{n_{r_j}}(t)=\beta(t)$ $(t\in[-\lambda,\lambda])$.
$\lim_{j\to\infty}h_{n_{r_j}(t)}=\lim_{j\to\infty}V(h_{n_{r_j}};-\lambda,t)-(V(h_{n_{r_j}};-\lambda,t)-h_{n_{r_j}}(t))=\alpha(t)-\beta(t)\doteq h(t)$.
Let $[-\lambda,\lambda]=[-n,n]$ and then use Cantor's diagonal argument. □

Remark 5. In order to obtain a unique spectral matrix, boundary conditions must be added at the end point where L is in the limit-circle case [Coddington–Levinson [7, p.251, l.17–l.19]]. See [Coddington–Levinson [7, p.243, l.2–l.3; l.6–l.7; p.244, l.5–l.6]].

Remark 6. Since constant points form an open set, the spectrum is a closed set [Coddington–Levinson [7, p.252, l.15]].

Remark 7. Coddington–Levinson [7, pp.256–257, Problem 6].

Proof. Let $f(t) = \psi(t, \tilde{\lambda})$.

$$\int_0^\infty \psi(t,\tilde{\lambda})\psi(t,\lambda)dt = g(\lambda) = \begin{cases} \frac{1}{\tilde{r}} & \text{if } \lambda = \tilde{\lambda} \\ 0 & \text{if } \lambda \neq \tilde{\lambda}. \end{cases}$$

$\int_0^\infty |\psi(t,\tilde{\lambda})|^2 dt = \int_{-\infty}^\infty |g(\lambda)|^2 d\rho(\lambda) = \frac{1}{\tilde{r}}$. □

Remark 8. Coddington–Levinson [7, p.257, Problem 7].

Proof. $h_{\delta n} \in \mathfrak{L}^2(0,\infty)$. Let $\delta \to (-\infty,\infty)$. □

Remark 9. By Coddington–Levinson [7, p.227, (2.11)], $\chi_{-\infty} \in \mathfrak{L}^2(-\infty,0)$. Consequently, $\cos\sqrt{l}t + m_{-\infty}(l)\frac{\sin\sqrt{l}t}{\sqrt{l}} = c(\cos\sqrt{l}t - i\sin\sqrt{l}t)$ for some constant c [Coddington–Levinson [7, p.252, l.-6–l.-5]].

Remark 10. $d\rho_{11}(\lambda) = 0\,(\lambda < 0)$ [Coddington–Levinson [7, p.253, l.5]].

Proof. $\frac{i}{2\sqrt{\nu+i\varepsilon}} = \frac{i\sqrt{\nu-i\varepsilon}}{2\sqrt{\nu^2+\varepsilon^2}}$.
If $\nu < 0$, then $\Im(\frac{i\sqrt{\nu}}{2|\nu|^2}) = 0$. □

Remark 11. Coddington–Levinson [7, pp.254–255, Problem 1].

Proof. (a) ψ_λ and $\psi'_\lambda \to \infty$, $-\infty$, or 0 as $t \to \infty$.

I ψ_λ can have at most one zero for $t > t_0$.

Proof. Let t_1, t_2 be consecutive zeros of ψ_λ on (t_0,∞).
Suppose $\psi_\lambda(t) > 0$ on (t_1,t_2). Then $\psi''_\lambda(t) > 0$ on (t_1,t_2).
Namely, $\psi'_\lambda(t)$ is increasing on (t_1,t_2).
$\psi'_\lambda(t_1) \geq 0$ because $\psi_\lambda(t_1) = 0$ and $\psi_\lambda(t) > 0$ on (t_1,t_2).
$\psi'_\lambda(t_1) > 0$ because $\psi_\lambda(t_1) = \psi'_\lambda(t_1) = 0$ would imply $\psi_\lambda(t) \equiv 0$ on (t_1,t_2).
Since $\psi'_\lambda(t) > 0$ on (t_1,t_2), $\psi_\lambda(t_2) > 0$. This contradicts $\psi_\lambda(t_2) = 0$. □

II ψ'_λ can have at most one zero for $t > t_0$.

Proof. Let t_1, t_2 be consecutive zeros of ψ'_λ on (t_0, ∞).
Suppose $\psi'_\lambda(t) > 0$ on (t_1, t_2). Then ψ_λ is increasing on (t_1, t_2).
Case $\psi_\lambda(t_1) > 0$:
Then $\psi_\lambda(t) > 0$ on (t_1, t_2). Hence $\psi''_\lambda(t) > 0$ on (t_1, t_2).
Since $\psi'_\lambda(t_1) = 0$, $\psi'_\lambda(t_2) > 0$. This contradicts $\psi'_\lambda(t_2) = 0$.
Case $\psi_\lambda(t_2) < 0$:
Then $\psi_\lambda(t) < 0$ on (t_1, t_2). Hence $\psi''_\lambda(t) < 0$ on (t_1, t_2).
Since $\psi'_\lambda(t_1) = 0$, $\psi'_\lambda(t_2) < 0$. This contradicts $\psi'_\lambda(t_2) = 0$.
Case $\psi_\lambda(t_1) < 0$ and $\psi_\lambda(t_2) > 0$:
There exists an zero t_3 of ψ_λ on (t_1, t_2). Take the largest zero t_3^* of ψ_λ on (t_1, t_2).
Then $\psi''_\lambda(t) > 0$ on $(t_3^*, t_2]$.
Since $\psi'_\lambda(t_3^*) > 0$, $\psi'_\lambda(t_2) > 0$. This contradicts $\psi'_\lambda(t_2) = 0$. □

III (i) If $(\exists t_1 > t_0 \ni \forall t \geq t_1,\ \psi'_\lambda(t) > 0)$ and $\psi_\lambda(\infty) = \infty$, then $\psi'_\lambda(\infty) = \infty$.

Proof. $\psi_\lambda(\infty) = \infty \Rightarrow \psi''_\lambda(\infty) = \infty$.
$\forall M > 0, \exists t_2 > t_1 \ni \forall t \geq t_2,\ \psi''_\lambda(t) \geq M$.
$\forall t \geq t_2 + 1,\ \psi'_\lambda(t) - \psi'_\lambda(t_2) = \psi''_\lambda(t^*)(t - t_2) \geq M$. □

(ii) If $(\exists t_1 > t_0 \ni \forall t \geq t_1,\ \psi'_\lambda(t) > 0$ and ψ_λ is bounded above) or if $(\exists t_1 > t_0 \ni \forall t \geq t_1,$ $\psi'_\lambda(t) < 0$ and ψ_λ is bounded below), then $\psi_\lambda(\infty) = \psi'_\lambda(\infty) = 0$.

Proof. Since $\psi'_\lambda(t) > 0$ [resp. < 0] on (t_1, ∞), $\psi_\lambda(t)$ is increasing [resp. decreasing] on (t_1, ∞).
Let $c = \psi_\lambda(\infty)$. Then $-\infty < c < \infty$ and $\psi'_\lambda(\infty) = 0$.
Fix $\varepsilon > 0$. $\exists t_2 \geq t_1 \ni \forall t \geq t_2, |\psi'_\lambda(t)| \leq \varepsilon$.
Case $c > 0$:
Then $\psi''_\lambda(\infty) = \infty$.
$\forall M > 0, \exists t_3 > t_2 \ni \forall t \geq t_3,\ \psi''_\lambda(t) \geq M$.
$\forall t \geq t_3 + 1,\ 2\varepsilon \geq \psi'_\lambda(t) - \psi'_\lambda(t_3) = \psi''_\lambda(t^*)(t - t_3) \geq M$, a contradiction.
Case $c < 0$:
Then $\psi''_\lambda(\infty) = -\infty$.
$\forall M > 0, \exists t_3 > t_2 \ni \forall t \geq t_3,\ \psi''_\lambda(t) \leq -M$.
$\forall t \geq t_3 + 1,\ -2\varepsilon \leq \psi'_\lambda(t) - \psi'_\lambda(t_3) = \psi''_\lambda(t^*)(t - t_3) \leq -M$, a contradiction. □

(iii) If $(\exists t_1 > t_0 \ni \forall t \geq t_1,\ \psi'_\lambda(t) < 0)$ and $\psi_\lambda(\infty) = -\infty$, then $\psi'_\lambda(\infty) = -\infty$.

Proof. $\psi_\lambda(\infty) = -\infty \Rightarrow \psi''_\lambda(\infty) = -\infty$.
$\forall M > 0, \exists t_2 > t_1 \ni \forall t \geq t_2,\ \psi''_\lambda(t) \leq -M$.
$\forall t \geq t_2 + 1,\ \psi'_\lambda(t) - \psi'_\lambda(t_2) = \psi''_\lambda(t^*)(t - t_2) \leq -M$. □

(b) I There cannot be two linearly independent eigenfunctions corresponding to the same eigenvalue.

Proof. Assume χ_1, χ_2 are two such eigenfunctions which satisfy
$\sin\alpha\chi(0) - \cos\alpha\chi'(0) = 0$. Then all the solutions of $Lx = \lambda x$ satisfy
$\sin\alpha\chi(0) - \cos\alpha\chi'(0) = 0$. This would contradict the existence of solution. □

II By 11(a)I and Hartman [17, p.326, (ii)], ψ_λ has a finite number of zeros on $(0, \infty)$.

III By Coddington–Levinson [7, p.212, l.-14–l.-13], the zeros of ψ_λ move continuously to the left as λ increases and to the right as λ decreases.

IV By Coddington–Levinson [7, p.212, (2.5) & (2.6)], ψ_λ has zeros on $(0,\infty)$ if λ is large enough.

V ψ_λ has no zeros on $(0,\infty)$ if λ is near $-\infty$.

Proof. (i) Fix $\delta_{\delta>0 \text{ and } \alpha<\pi-\delta}$.
$\exists t_1 \ni \forall t \geq t_1, q(t) > 0$.
Let $Q = \max_{t\in[0,t_1]} |q(t)|$.
By Coddington–Levinson [7, p.213, l.10–l.12],
$(\delta \leq \omega \leq \pi - \delta) \Rightarrow \omega' \leq 1 - |\lambda| \sin^2 \delta + Q$.

(ii) Fix $t^* > 0$. $\exists \lambda^* < 0 \ni \forall t \geq t^*, \forall \lambda < \lambda^*, \omega(t,\lambda) \leq \pi - \delta$.

Proof. Assume $\forall n > 0$, $\exists t_n \geq t^*$ and $\exists \lambda_{*n} \leq -n \ni \omega(t_n, \lambda_{*n}) > \pi - \delta$.
$\exists t_n^* \in (0,t_n) \ni \omega(t_n^*, \lambda_{*n}) = \pi - \delta$.
By 11(b)Vi, $\omega(t,\lambda_{*n}) = \pi - \delta \Rightarrow \omega'(t,\lambda_{*n}) < 0$ for λ_{*n} near $-\infty$.
Consequently, $\omega(t_n,\lambda_{*n}) \leq \pi - \delta$, a contradiction. □

(iii) Fix $t_* > 0$. $\exists \lambda_* < \lambda^* \ni \forall t \geq t_*, \forall \lambda \leq \lambda_*, \omega(t,\lambda) \leq \delta$.

Proof. Assume $\forall n > 0$, $\exists t_n \geq t^*$ and $\exists \lambda_{*n} \leq -n \ni \omega(t_n, \lambda_{*n}) > \delta$.
By 11(b)Vi, $\exists \lambda_{**} < \lambda^* \ni \forall \lambda < \lambda_{**}, \forall \omega_{\delta \leq \omega \leq \pi - \delta}, \omega'(t,\lambda) < \frac{-10}{t_*}$.

$$\begin{aligned} 0 - (\pi - \delta) &\leq \omega(t_n, \lambda_{*n}) - \omega(0, \lambda_{*n}) \quad \text{[Coddington–Levinson [7, p.212, l.}-6\text{]]} \\ &= \omega'(t_n^*, \lambda_{*n}) t_n < -10, \text{ a contradiction.} \end{aligned}$$

□

(iv) $\forall t \geq t_*$, $\lambda \leq \lambda_*$, $0 \leq \omega(t,\lambda) \leq \delta$ [Coddington–Levinson [7, p.212, l.-6] and 11(b)Viii].
If ψ_λ always has zeros on $(0,\infty)$ as $\lambda \to -\infty$, then
$\exists t^* \geq t_*, \exists \lambda \leq \lambda_* \ni \omega(t^*,\lambda) = 0$.
By Coddington–Levinson [7, p.212, (2.4)], $\omega' > 0$ whenever $\omega = 0$.
Consequently, $\exists \eta > 0 \ni \omega(t^* - \eta, \lambda) < 0$, a contradiction.

□

VI $|\psi(t,\lambda)| \to \infty$ as $t \to \infty$, $\lambda \to -\infty$.

Proof. Fix $\delta > 0$ and $t_1 > 0$.
By 11(b)Viii, $\exists \lambda_* < \lambda^* \ni \forall t \geq t_1, \forall \lambda \leq \lambda_*, \omega(t,\lambda) \leq \delta$.
By 11(b)V, $\omega(t,\lambda) > 0$.
Hence, $\psi(t,\lambda)$ and $\psi'(t,\lambda)$ have the same sign.
Let $\psi(t_1,\lambda_*) > 0$. By 11(b)V, $\forall t \geq t_1, \forall \lambda \leq \lambda_*, \psi(t,\lambda) > 0$.
By 11a, we may divide the discussion into the following three cases:
If both $\psi(t,\lambda)$ and $\psi'(t,\lambda)$ approach ∞ or $-\infty$ as $t \to \infty$, the proof is trivial.
Now suppose both $\psi(t,\lambda)$ and $\psi'(t,\lambda)$ approach 0 as $t \to \infty$.
$\exists t_2 > t_1 \ni 0 < \psi(t_2,\lambda) < \psi(t_1,\lambda)$.
$\psi(t_2,\lambda) - \psi(t_1,\lambda) = \psi'(t^*,\lambda)(t_2 - t_1) \Rightarrow \psi'(t^*,\lambda) < 0$, a contradiction. □

VII $\inf\{\lambda | \psi_\lambda$ has at least $n+1$ zeros on $(0,\infty)\}$
$= \sup\{\lambda | \psi_\lambda$ has at most n zeros on $(0,\infty)\}$
$\doteq \lambda_n$.

(c) Given any $\varepsilon > 0$ and n, $\lambda_{bn} < \lambda_n + \varepsilon$ for b large enough.

Proof. ψ_{bn} has $n+1$ zeros on $(0,b]$ and $\psi_{bn}(b) = 0$.
When $\varepsilon > 0$ is small enough, $\psi_{\lambda_n+\varepsilon}$ has exactly $n+1$ zeros on $(0,\infty)$.
Let t_{n+1} be the $(n+1)^{st}$ zero of $\psi_{\lambda_n+\varepsilon}$ and let $b > t_{n+1}$. □

(d) The $\psi_k \in \mathfrak{L}^2(0,\infty)$ are orthogonal.

Proof. ψ_k, $\psi_k' \to 0$ as $t \to \infty$.
Suppose $\psi_k > 0$ on (t_k,∞), where t_k is the k^{th} zero of ψ_k on $(0,\infty)$.
Since $\psi_k'' > 0$ on (t_k,∞), ψ_k' is increasing on (t_k,∞).
$\int_{t_k}^\infty |\psi_k''| dt = \int_{t_k}^\infty \psi_k'' dt = \psi_k'(\infty) - \psi_k'(t_k) = -\psi_k'(t_k) > 0$.
Suppose $\psi_k < 0$ on (t_k,∞). Since $\psi_k'' < 0$ on (t_k,∞), ψ_k' is decreasing on (t_k,∞).
$\int_{t_k}^\infty |\psi_k''| dt = -\int_{t_k}^\infty \psi_k'' dt = -(\psi_k'(\infty) - \psi_k'(t_k)) = \psi_k'(t_k) > 0$.
$(\lambda_j - \lambda_k)\int_0^\infty \psi_j \psi_k dt = \int_0^\infty (\psi_k L\psi_j - \psi_j L\psi_k) = [\psi_j\psi_k](\infty) - [\psi_j\psi_k](0) = 0$. □

□

Remark 12. By Coddington–Levinson [7, p.244, l.7–l.10], the functions $m_{-\infty}$, m_∞ are meromorphic with simple poles on the real axis [Coddington–Levinson [7, p.253, l.-9–l.-8]].

Remark 13. $m_{-\infty}(l) - m_\infty(l)$ has only isolated zeros [Coddington–Levinson [7, p.253, l.-7]].

Proof. Let l_0 be a zero of $m_\infty - m_{-\infty}$, but not an isolated zero.
Then $[\frac{d}{dl}(m_\infty - m_{-\infty})](l_0) = 0$. This implies
$m_\infty(l_0) - m_{-\infty}(l_0) = a_0(l-l_0)^2[\cdots]$ (*),
where $a_0 \neq 0$.

$$\begin{aligned} \Im l > 0 &\Rightarrow (\Im m_\infty > 0 \text{ and } \Im m_{-\infty}(l) < 0) \quad \text{[Coddington–Levinson [7, p.250, l.8]]} \\ &\Rightarrow \Im(m_\infty(l) - m_{-\infty}(l)) > 0. \end{aligned}$$

This would contradict (*). □

Remark 14. $\{\lambda_n\}, n = 1,2,\ldots$, given in Coddington–Levinson [7, p.253, l.-5] are zeros of $m_{-\infty}(l) - m_\infty(l)$.

Remark 15. The equality $\lim_{|t|\to\infty} \frac{\log|u_j|}{t^2} = 1$ given in Coddington–Levinson [7, p.254, l.5] should have been corrected as $\lim_{|t|\to\infty} \frac{\log|u_j|}{2t^2} = 1$.

Proof. Let l be real.
There exists an even number $N > \max([\frac{l-1}{2}],0)$ such that $(k \geq N \Rightarrow \max(0, \frac{2-\varepsilon}{k}) \leq \frac{a_{k+2}}{a_k} \leq \frac{2+\varepsilon}{k})$.
Let $N = 2k_0$.
Assume $a_N \geq 0$.
$a_N t^{2k_0} e^{(2-\varepsilon)t^2} \leq a_N t^{2k_0} \sum_{k=0}^\infty (\frac{a_{2(k_0+k)}}{a_N} t^{2k}) \leq a_N t^{2k_0} e^{(2+\varepsilon)t^2}$.
Let $u_1^*(t) = a_N t^{2k_0} \sum_{k=0}^\infty (\frac{a_{2(k_0+k)}}{a_N} t^{2k})$. Then

$\lim_{t\to\infty}\frac{|u_1^*(t)|}{e^{2t^2}}=1$. Hence
$\lim_{t\to\infty}\frac{|u_1(t)|}{e^{2t^2}}=1$. Similarly, $\lim_{t\to\infty}\frac{|u_2(t)|}{e^{2t^2}}=1$. □

Remark 16. $(c_1u_1+c_2u_2)e^{-\frac{1}{2}t^2}$ is unbounded for all c_1, c_2 that satisfy $c_1^2+c_2^2\neq 0$ [Coddington–Levinson [7, p.254, l.6]].

Proof. If c_1 or c_2 is 0, then the proof is trivial.
Case $\lim_{t\to\infty}\frac{u_1(t)}{|u_1(t)|}=\lim_{t\to\infty}\frac{u_2(t)}{|u_2(t)|}=1$:
If $c_1\neq -c_2$, then
$\lim_{t\to\infty}\frac{(c_1u_1+c_2u_2)(t)}{e^{2t^2}}=c_1+c_2\neq 0$.
If $c_1=-c_2\neq 0$, then
$\lim_{t\to-\infty}\frac{(c_1u_1+c_2u_2)(t)}{e^{2t^2}}$
$=c_1[\lim_{t\to\infty}\frac{u_1(t)}{e^{2t^2}}+\lim_{t\to\infty}\frac{u_2(t)}{e^{2t^2}}]$ (since u_2 is odd)
$=2c_1$. □

8 Dualty of adjoint systems

[Coddington–Levinson [7, chap. 11]; Ince [18, chap. IX]]

Theorem 8.1. Coddington–Levinson [7, p.289, Theorem 3.1]

Corollary. A necessary and sufficient condition that a linear differential system is self-adjoint [Coddington–Levinson [7, p.291, Theorem 3.2]; Ince [18, §9·4; §9·41]]

Theorem 8.2. Coddington–Levinson [7, p.292, Theorem 3.4]

Theorem 8.3. Coddington–Levinson [7, p.294, Theorem 4.1]

Theorem 8.4. Coddington–Levinson [7, p.296, Theorem 4.2]

Remark 1. The similarity between a linear differential equation and a system of linear algebraic equations [Ince [18, p.205, l.−17; l.−10–l.−8; l.−5]]; the similarity between boundary conditions and a system of linear algebraic equations [Ince [18, p.206, l.7; l.9; l.11]]

Remark 2. The incompability of a homogeneous [differential or algebraic] system means that the dimension of the null space [Jacobson [20, vol. 2, p.44, l.−1]] of (U) [Ince [18, p.208, l.7]] or$A$ [Ince [18, p.207, l.13]] is 0. The incompability of a non-homogeneous [differential or algebraic] system means that there are no solutions [Ince [18, p.208, l.−13–l.−11; p.207, l.−21–l.−19]].

Remark 3. By Jacobson [20, vol. 2, p.45, Theorem 4], there will be $n-p$ linearly independent sets of values $c_1,\cdots,c_n$ and corresponding to each of these sets of values there will be one solution of the differential equation which satisfies the boundary conditions [Ince [18, p.208, l.10–l.13]].

Remark 4. (Conventions)

In order to prove the necessary condition of the statement given in Ince [18, p.208, l.−9–l.−7], we should assume that m is the rank of (U); see Coddington–Levinson [7, p.286, l.−2–l.−1]. Coddington–Levinson [7, p.285, l.−5–l.−4] should have mentioned that $\det S\neq 0$; see Ince [18, p.209, l.12–l.13].

Remark 5. The statement "But this is precisely the matrix which determines the index of the adjoint system." given in Ince [18, p.213, l.-16–l.-15] should have been corrected as follows:
But this is the transpose of (V), where (V) determines the index of the adjoint system.
Remark. If we fail to correct the mistake in this specific case by using Coddington–Levinson [7, p.293, l.-11], we may easily get lost in proving the general case [Ince [18, p.213, l.-6]].

Remark 6. A necessary and sufficient condition that the complete system

$$(A) \quad \begin{cases} L(u) = r \\ U_i(u) = \gamma_i \qquad (i = 1, 2, \cdots, m) \end{cases}$$

shouuld have a solution is that every solution v of the homogeneous adjoint system

$$(B) \quad \begin{cases} \bar{L}(v) = 0 \\ V_i(v) = 0 \qquad (i = 1, 2, \cdots, 2n - m) \end{cases}$$

satisfies the relation
$(C) \quad \int_a^b vrdx = \gamma_1 V_{2n}(v) + \cdots + \gamma_m V_{2n-m+1}(v)$.

Proof. Let k' be the index of the homogeneous system (B).
I. Case $k' = 0$.
1. $k' = 0 \Rightarrow$ the index of the reduced system of (A) is $n - m$ [Ince [18, p.214, l.-11–l.-3]]
$\Rightarrow$ the rank of (U) is m [Ince [18, p.208, l.7–l.8]].
2. $k' = 0 \Leftrightarrow$ the index of the reduced system of (A) is $n - m$ [Ince [18, p.214, l.-11–l.-3]]
$\Leftrightarrow$ the non-homogeneous system (A) has a solution [Ince [18, p.208, l.-9–l.-7]].
Note that the proof of $\Rightarrow$ part requires the conclusion of 1.
Under the condition $k' = 0$, $v = 0$. Thus, $V_{2n}(v) = \cdots = V_{2n-m+1}(v) = 0$. Consequently, (C) holds.
II. Case $k' > 0$.
Let $v_1, \cdots, v_{k'}$ form a linearly independent set of solutions of the system (B).
$\Rightarrow$: (C) follows from Green's formula [Ince [18, p.212, l.4]].
$\Leftarrow$: 1. Let u_0 be any solotion of the equation $L(u) = r$.
By Green's formula [Ince [18, p.212, l.4]], $\int_a^b vrdx = U_1(u_0)V_{2n}(v) + \cdots + U_m(u_0)V_{2n-m+1}(v)$.
By (C), it follows that
$(D) \quad \{\gamma_1 - U_1(u_0)\}V_{2n}(v) + \cdots + \{\gamma_m - U_m(u_0)\}V_{2n-m+1}(v) = 0$.
Let $u_1, \cdots, u_n$ be a fundamental system of solutions of the homogeneous equation $L(u) = 0$. Then by Green's formula [Ince [18, p.212, l.4]],
$(E) \quad (U)^T \begin{pmatrix} V_{2n}(v) \\ \vdots \\ V_{2n-m+1}(v) \end{pmatrix} = \hat{0}$, where$(U) = \begin{pmatrix} U_1(u_1) & \cdots & U_1(u_n) \\ \vdots & \ddots & \vdots \\ U_m(u_1) & \cdots & U_m(u_n) \end{pmatrix}$. Thus,
$(F) \quad (U^1)^T \begin{pmatrix} V_{2n}(v) \\ \vdots \\ V_{2n-m+1}(v) \end{pmatrix} = \hat{0}$, where$(U^1) = \begin{pmatrix} U_1(u_1) & \cdots & U_1(u_n) & \gamma_1 - U_1(u_0) \\ \vdots & \ddots & \vdots & \vdots \\ U_m(u_1) & \cdots & U_m(u_n) & \gamma_m - U_m(u_0) \end{pmatrix}$.
By Ince [18, p.212, l.-14], $V_{2n}(v_i), \cdots, V_{2n-m+1}(v_i) (i = 1, \cdots, k')$ are linearly independent solutions of the equation (F).
2. The rank of $(U^1)^T$ is therefore at most $m - k'$, but

rank $(U)^T = $ rank $(U) = n-k$
$= m-k'$ [Ince [18, p.213, l.-7]]. Consequently, rank $(U^1)^T =$ rank $(U)^T$. Therefore, the last row $(\gamma_1 - U_1(u_0), \cdots \gamma_m - U_m(u_0))$ of $(U^1)^T$ is a linear combination of the first n rows $(U_1(u_i), \cdots, U_m(u_i) (i = 1, \cdots, n)$. □

Remark 7. The "linear subsets" said in Coddington–Levinson [7, p.289, l.9] refers to "a subspace of a vector space".

Remark 8. By Coddington–Levinson [7, p.288, l.11–l.12], D^+ is uniquely determined by U, although U^+ is not [Coddington–Levinson [7, p.289, l.11]].

Remark 9. $[xy](b) - [xy](a) = Ux \cdot U_c^+ y + U_c x \cdot U^+ y$ [Coddington–Levinson [7, p.288, l.8, (2.5)]].

Proof. $[xy](b) - [xy](a) = S(f,g)$ [Coddington–Levinson [7, p.288, l.13–l.16]]
$= Hf \cdot Jg$ [Coddington–Levinson [7, p.287, l.-15]]
$= f^t H^t \bar{J} \bar{g}$. □

Remark 10. If $\tilde{U}_c$ is any other complementary form to U, and $\tilde{U}_c^+, \tilde{U}^+$ the corresponding forms of rank m and $2n-m$, then $\tilde{U}^+ y = C^* U^+ y$ for some nonsingular matrix C [Coddington–Levinson [7, p.288, l.9–l.11]].

Proof. Let $j = m, k = 2n$.
$\tilde{H} f \cdot \tilde{J} g = Hf \cdot Jg$ [Coddington–Levinson [7, p.287, l.-8]].
$\tilde{J} g = \begin{pmatrix} G_m & G_- \\ 0_- & G_{2n-m} \end{pmatrix} Jg$ [Coddington–Levinson [7, p.288, l.3]].
Consequently, $\tilde{U}^+ y = C^* U^+ y$, where $C^* = G_{2n-m}$. □

Remark 11. $(Lu, v) = (u, L^+ v)$ for all $u \in C^n$ on $[a,b]$ satisfying (3.1) and all $v \in C^n$ on $[a,b]$ satisfying (3.2) [Coddington–Levinson [7, p.289, l.7–l.8]].

Proof. $(Lu, v) - (u, L^+ v) = \int_a^b (Lu) \bar{v} dt - \int_a^b u \overline{L^+ v} dt$ [Coddington–Levinson [7, p.289, l.6]]
$[uv](b) - [uv](a)$ [Coddington–Levinson [7, p.284, l.-3, (1.2)]]
$= Ux \cdot U_c^+ y + U_c x \cdot U^+ y$ [Coddington–Levinson [7, p.288, l.8, (2.5)]]. □

Remark 12. If U^+ is determined as in Coddington–Levinson [7, p.288, Theorem 2.1] and P, Q are defined as in Coddington–Levinson [7, p.289, l.17], then (3.3) holds. Conversely, if P, Q are defined as in Coddington–Levinson [7, p.289, l.17] and (3.3) holds, then $U^+ x = 0$ is an adjoint boundary condition to (3.1) [Coddington–Levinson [7, p.289, Theorem 3.1]].

Remark 13. Since $H_1 = HA$ and the $2n-m$ columns of H and those of H_1 are two bases of the null space of $(M:N)$, A is nonsingular [Coddington–Levinson [7, p.290, l.-2]].

Remark 14. $U_1^+ y = A^* U^+ y$ [Coddington–Levinson [7, p.291, l.1]].

Proof. $U_1^+ y = P_1^* \eta(a) + Q_1^* \eta(b)$ [Coddington–Levinson [7, p.290, l.11]]
$= (PA)^* \eta(a) + (QA)^* \eta(b)$ [Coddington–Levinson [7, p.290, l.1]]
$= A^* (P^* \eta(a) + Q^* \eta(b))$
$= A^* U^+ y$ [Coddington–Levinson [7, p.290, l.-8]]. □

Remark 15. π_m is the adjoint problem of π^+_{2n-m} [Coddington–Levinson [7, p.291, l.9]].

Proof. I. $(L^+)^+ = L$.

Proof using the Lagrange identity. See Ince [18, p.125, l.6; l.20]. □

Proof using the symmetry relation between adjoint differential systems. Using the one-to-one correspondence between their solutions [Coddington–Levinson [7, p.82, l.−9–l.−8]], we may identify the differential equation given in Coddington–Levinson [7, p.82, l.1–l.2] with the differential system given in Coddington–Levinson [7, p.82, (6.1); (6.2)].
By Coddington–Levinson [7, p.85, l.−13–l.−4], from a solution of the differential system given in Coddington–Levinson [7, p.85, (6.8); (6.9)] we may determine the differential operator L^+_n and find a solution of the differential equation given in Coddington–Levinson [7, p.85, l.−5]. Conversely, we may find a solution of the differential system given in Coddington–Levinson [7, p.85, (6.8); (6.9)] from a solution of the differential equation given in Coddington–Levinson [7, p.85, l.−5]. This one-to-one correspondence allows us to identify the differential equation given in Coddington–Levinson [7, p.85, l.−5] with the differential system given in Coddington–Levinson [7, p.85, (6.8); (6.9)]. However, under this identification the differential operator is determined up to a factor $(-1)^n$ [compare Coddington–Levinson [7, p.85, l.−7] with Coddington–Levinson [7, p.85, l.−5]]; the choice of sign given in the definition of adjoint differential operator [Coddington–Levinson [7, p.84, l.−6]] is arbitrary.
Consequently, the result follows from the statement given in Coddington–Levinson [7, p.71, l.2–l.3]. □

II. $\int_a^b [(Lu)\bar{v} - u\overline{L^+v}]dt = [xy](b) - [xy](a)$ [Coddington–Levinson [7, p.86, Corollary]]
$= Ux \cdot U_c^+ y + U_c x \cdot U^+ y$ [Coddington–Levinson [7, p.288, Theorem 2.1]].
Let new L be L^+. $\int_a^b [(\overline{L^+v})\bar{u} - v(\overline{Lu})]dt = -\overline{Uy} \cdot \overline{U_c^+ x} - \overline{U_c y} \cdot \overline{U^+ x}$.
By Coddington–Levinson [7, p.288, Theorem 2.1], $-\overline{Uy} = 0$ is an adjoint boundary condition to $\overline{U^+x} = 0$. Namely,
$Uy = 0$ is an adjoint boundary condition to $U^+x = 0$. □

Remark. $U_r\Delta_{r-1} = U_{r-1}\Delta'_r - U'_{r-1}\Delta_r$ [Ince [18, p.120, l.−2]].

Proof. The two sides of the equality are linear differential operators of order r whose coefficient of $u^{(r)}$ is $(-1)^r\Delta_{r-1}\Delta_r$. Since $u_1, \cdots, u_r$ are solutions of the corresponding homogeneous differential equations of both operators, by Coddington–Levinson [7, p.70, l.−9], the two differential operators are equal. □

Remark 16. (If the original proof is essentially straightford, we should avoid giving it an outlook of a reduction to absurdity) We should not make things unnecessarily complicated. The proof given in Ince [18, p.212, l.10–p.213, l.3] is straightforward if we eliminate the first four wards and the last sentence. Coddington–Levinson [7, p.292, l.−7] follows Ince's style by using an unnecessary reduction to absurdity. The proof given in Ince [18, p.213, l.−7–−4] does not use reduction to absurdity. However, Coddington–Levinson [7, p.293, l.−7] gives this argument of Ince's the unnecessary outlook of reduction to absurdity again.
Remark. Although the equality given in Ince [18, p.212, l.4] is the same as that given in

Coddington–Levinson [7, p.288, (2.5)], the latter form may make it easier to recognize the structural relationship between adjoint boundary-value problems.

Remark 17. The $U\Phi$ given in Coddington–Levinson [7, p.291, l.−2] is the same as the (U) defined in Ince [18, p.208, l.7].

Proof. $U\Phi = (M:N)\begin{pmatrix}\Phi(a)\\ \Phi(b)\end{pmatrix}$ [by definition]
$= (U)$ [Ince [18, p.211, l.−7–l.−6]]. □

Remark. The reason that Coddington–Levinson [7, p.291, l.−2] uses the notation $U\Phi$: unless we use Φ the meaning of the right side of the the identity given in Ince [18, p.208, l.7] cannot be complete. The reason that Ince [18, p.208, l.7] uses the notation (U): rank $(U\Phi)$ remains the same for any fundamental matrix Φ; see Coddington–Levinson [7, p.292, l.9].

Remark 18. Let A be a matrix and b a vector. Then $Ax = b$ has a solution if and only if $b \cdot u = 0$ for every solution u of $A^*x = 0$ [Coddington–Levinson [7, p.294, l.13–l.14]].

Proof. $\Rightarrow$: $\exists x : Ax = b$.
$b \cdot u = Ax \cdot u = (Ax, u) = (x, A^*u) = (x, 0) = 0$.
$\Leftarrow$: Let A be an $m \times n$ matrix, and $A : \mathbb{C}^n \to \mathbb{C}^m$. Then $A^* : \mathbb{C}^m \to \mathbb{C}^n$.

$$A^*u = 0 \Leftrightarrow \begin{pmatrix}\overline{a_{11}} & \cdots & \overline{a_{m1}}\\ \vdots & \ddots & \vdots\\ \overline{a_{1n}} & \cdots & \overline{a_{mn}}\end{pmatrix}\begin{pmatrix}u_1\\ \vdots\\ u_m\end{pmatrix} = 0$$

$$\Leftrightarrow \sum_{i=1}^m u_i\overline{a_{ij}} = 0 \quad (j = 1, \cdots, n)$$

$$\Leftrightarrow (u, \begin{pmatrix}a_{1j}\\ \vdots\\ a_{mj}\end{pmatrix}) = 0 \quad (j = 1, \cdots, n).$$

Thus, (the null space of A^*) $= (A\mathbb{C}^n)^\perp$, where $\perp$ refers to the scalar product on $\mathbb{C}^m$.
By hypothesis, $b \in$ (the null space of $A^*)^\perp = ((A\mathbb{C}^n)^\perp)^\perp$
$= A\mathbb{C}^n$ [Jacobson [20, vol. 2, p.151, l.17]]. Consequently,
$\exists x \in \mathbb{C}^n : b = Ax$. □

Remark 19. Similar reasoning shows that this is true for $\tau_1 > \tau_2$ [Coddington–Levinson [7, p.296, l.−7]].

Proof. In case $\tau_1 > \tau_2$, we still define G_1, G_2^+ as in Coddington–Levinson [7, p.296, l.9].
However, we consider the three intervals "$[a, \tau_2 - 0], [\tau_2 + 0, \tau_1 - 0], [\tau_1^+ 0, b]$" instead of "$[a, \tau_1 - 0], [\tau_1 + 0, \tau_2 - 0], [\tau_2^+ 0, b]$".
Then we still have (4.9). Following the argument given in Coddington–Levinson [7, p.296, l.−17–l.−8], we obtain $\bar{G}^+(\tau_1, \tau_2) - G(\tau_2, \tau_1) = 0$ for $\tau_1 > \tau_2$. □

9 Miscellaneous notes

(1) We may generalize Euler's solutions to the two differential equations given in Watson [41, p.62, l.−8–l.−2] as follows:

Let $y = x^{1/2}u$ and $z = 2na^{1/2}x^{1/(2n)}$. Then the following two differential equations are equivalent:
$x^{3/2}\frac{d^2y}{dx^2} + ax^{(n-2)/(2n)}y = 0$;
$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - n^2)u = 0$.

(2) (An advantaneous viewpoint can facilitate the calculations in a proof)
The proof [Pontryagin [27, p.50, l.-22–p.52, l.-4]] of the first part of Pontryagin [27, p.52, (B)] originates from the viewpoint of differentiation [Pontryagin [27, p.50, l.-14]]. In contrast, the proof [Hartman [17, p.324, l.1–l.16]] of Hartman [17, p.324, (1.15)(i)] originates from the viewpoint of changing variables [Hartman [17, p.324, l.11]]. The latter viewpoint can facilitate the calculations in the proof of Pontryagin [27, pp.50–51, Theorem 5].

(3) (Method vs. calculation for solutions) $u(t)$ and $v(t)$ are linearly independent solutions of (2.1) if and only if $c \neq 0$ in (2.7) [Hartman [17, p.327, l.8–l.9]]

Proof with theory as a guide. $c \neq 0$
$\Leftrightarrow \det X(t) \neq 0$ [Hartman [17, p.326, (2.7)]]
$\Leftrightarrow ((u(t), p(t)u'(t))$ and $(v(t), p(t)v'(t))$ are linearly independent)
$\Leftrightarrow (u(t)$ and $v(t)$ are linearly independent) [Hartman [17, p.326, (iv)]].
□

Proof with calculations in mind. See Ince [18, p.116, l.14–p.118, l.16]. □

Remark. The first proof helps us grasp the key points, while the second proof helps us understand the original approach. The second proof requires patience and tricks [Hartman [17, p.326, l.12–l.13]] to deal with nuisances [Ince [18, p.117, l.14]; signs of cofactors] and difficult points [Ince [18, p.117, l.-20]; continuity].

(4) (Lommel's formula) The solutions of $\Pi_{r=0}^{n-1}(\vartheta + \alpha - 2\gamma\beta)(\vartheta + \alpha - 2\beta\nu - 2\gamma\beta)u = (-)^n\beta^{2n}c^{2n}z^{2n\beta}u$ are of the form $u = z^{\beta\nu - \alpha}\mathscr{C}_\nu(\gamma z^\beta)$, where $\gamma = c\exp(r\pi i/n) \quad (r = 0, 1, \cdots, n-1)$ [Watson [41, p.107, l.13–l.16]].

Proof. Since $(\vartheta + a)(\vartheta + b) = (\vartheta + b)(\vartheta + a)$, it suffices to prove
$(\vartheta + \alpha - 2n\beta)(\vartheta + \alpha - 2\beta\nu - 2n\beta)(z^{2n\beta}u) = -\beta^2\gamma^2z^{2(n+1)\beta}u$.
$(\vartheta + \alpha - 2n\beta)(\vartheta + \alpha - 2\beta\nu - 2n\beta)z^{2n\beta}u$
$= [z^2\frac{d^2}{dz^2} + (1 + 2\alpha - 4n\beta - 2\beta\nu)z\frac{d}{dz} + (\alpha - 2n\beta)(\alpha - 2\beta\nu - 2n\beta)](z^{2n\beta}u)$
$= z^{2n\beta}[z^2\frac{d^2u}{dz^2}] + (1 + 2\alpha - 2\beta\nu)z\frac{du}{dz} + (\alpha^2 - 2\alpha\beta\nu)u]$
$= z^{2n\beta}[-\beta^2\gamma^2z^{2\beta}]u$ [Watson [41, p.97, (3)]]. □

Application. The above theorem can be used to solve $\frac{d^2}{dz^2}\{z^{4\mu}\frac{d^2u}{dz^2}\} = z^{2\mu}u$ for $\mu = 0, 1, 4/5, 3$ [Watson [41, p.107, l.19–p.108, l.9]].
Remark. For r in Watson [41, p.107, l.16], by Guo–Wang [16, p.350, (18)], it is unnecessary to take $r = n, n+1, \cdots, 2n-1$.

(5) If Riccati's equation $\frac{dy}{dx} = az^n + by^2$ is integrable in finite terms, then $n = -2$ or $-\frac{4m}{2m\pm1} \quad (m = 0, 1, 2, \cdots)$ [Watson [41, p.123, l.23–l.26]].

Remark 1. $n \leq \mu \leq m$ [Watson [41, p.112, l.11]].

Proof. Let $\mu+1$ be the order of $\psi_{p,q}(z)$ [Watson [41, p.114, l.8]].
By Watson [41, p.114, l.10], $\mu+1 \leq m+1$.
By Watson [41, p.112, (1)], $\mu+1 \geq n$.
If $\mu+1=n$, then (1) has a solution of order n [Watson [41, p.114, l.11]].
If $\mu+1>n$, then $\mu \geq n$. □

Remark 2. $\theta = lf_m(z), \vartheta = \varsigma f_m(z)$, and $\Theta = ef_m(z)$ [Watson [41, p.112, l.15–l.16]].

Remark 3. By Pontryagin [27, p.51, (3)], $F = e^{\alpha\theta}\{\Phi_1+\Phi_2\theta\}$ [Watson [41, p.113, l.−9]].

Remark 4. If $G_{\Theta,\Theta} = AG\Theta + BG$, then, by Collatz [9, p.111, l.12–l.13], $G = \Theta^{\gamma}\{\Phi_1+\Phi_2 f_\mu(z)\}$ [Watson [41, p.115, l.12]].

Remark 5. By Collatz [9, p.111, (11.62)], γ and δ are the roots of the equation $(x(x-1)-Ax-B=0$ [Watson [41, p.115, l.15–l.16]].

Remark 6. t is of order μ at most [Watson [41, p.116, l.12]] because if $u_1 = ef_\mu(z)$, then $\log u_1 = f_\mu(z)$ is of order μ.

Remark 7. By Jackson [19, p.126, l.4–l.5], the Wronskian of any two solutions of (1) is a constant [Watson [41, p.117, l.1]].

Remark 8. "u is an algebraic integral of (1)" means "u is a solution of (1) and u is algebraic over the field $\mathbb{C}(z)$".

Remark 9. If not, all the roots of (3) would be zero [Watson [41, p.118, l.−2]].

Proof. By Jacobson [20, vol. 1, p.110, Ex.4], $p_1 = \cdots = p_M = 0$.
By Jacobson [20, vol. 1, p.108, l.5], $\mathscr{A}(u,z) = u^M$. □

Remark 10. $\lambda A_\lambda . s! \frac{2.4.6\cdots s}{1.3.5\cdots(s-1)} = 0$ follows from Watson–Whittaker [40, p.282, l.−1] and Guo–Wang [16, p.99, l.2].

Remark 11. M must be even [Watson [41, p.120, l.6]].

Proof. Assume M were odd. By Watson [41, p.120, l.1] and Jacobson [20, vol. 1, p.110, Ex.4], $p_1 = p_3 = \cdots = p_M = 0$.
$\mathscr{A}(u,z) = u^M + u^{M-2}p_2 + \cdots + up_{M-1}$. $\mathscr{A}$ contains a factor u, so $\mathscr{A}$ cannot be irreducible. □

Remark 12. The constant terms in the functions $\mathfrak{B}_r$ are not all zero [Watson [41, p.120, l.14]].

Proof. If for every $r \in \{1,2,\cdots,M/2\}$, the constant term in $\mathfrak{B}_r$ were zero, then for every $m \in \{m=1,\cdots,M\}$, the constant term in p_m would be zero. By Jacobson [20, vol. 1, p.110, Ex.4], for every $r \in \{1,2,\cdots,M/2\}$, the constant term in s_{2r} would be zero. This would contradict the statement given in Watson [41, p.120, l.4]. □

Remark 13. If Bessel's equation has an integral expressible in finite terms, then (2) must have a solution which is of order zero [Watson [41, p.121, l.1]].
Remark. By Watson [41, p.120, l.−14], the option [Watson [41, p.112, l.7]] that there exists a 0-order solution of Watson [41, p.120, l.−7, (1)] cannot occur.

Remark 14. An irreducible polynomial in $\mathbb{C}[z]$ has only simple roots [Van der Waerden [36, vol. I, p.120, l.1–l.2]], so [((3) has a pair of equal roots)⇒((3) is reducible)].

Remark 15. $\exists q : \kappa_q \neq 0$ [Watson [41, p.121, l.-2]].

Proof. Assume that $\forall q, \kappa_q = 0$.
By Watson [41, p.122, l.12], $\lambda = 0$. Thus, V becomes a constant. Similar to the argument given in Watson [41, p.122, l.13], we may prove that $\mathscr{A}$ is reducible, a contradiction. □

Remark 16. The $B_{1,q}$ are not all zero [Watson [41, p.122, l.-11–l.-10]].

Proof. If all the $B_{1,q}$ were zero, then $u = z^{-p}e^{\pm iz}$. That is, $w = 1$ [Watson [41, p.122, l.-4]]. However, $w = 1$ would not satisfy Watson [41, p.122, (7)]. Then u would not satisfy Watson [41, p.120, (1)]. Therefore, $B_{1,q}$ cannot be all zero. □

Summary. Since Riccati's equation is a variant [Ince [18, p.24, l.-11–p.25, l.4]; Watson [41, p.96, (6)]] of Bessel's equation, we may use the language of Bessel's functions to translate the above theorem as follows:
If Bessel's equation for functions of order ν [Watson [41, p.117, l.-10]] is soluble in finite terms, then 2ν is an odd integer.

In the proof [Watson [41, §4·7–§4·74]] of the latter version, despite possible difficulties, all the problems can be solved except one. That is, an infinite power series cannot be expressed as a polynomial [Watson [41, p.123, l.4]]. From this, we may determine the cases that Bessel's equation is soluble in finite terms [Watson [41, p.123, l.6–l.8]].

(6) The Ritz method is an effective tool for studying Sturm–Liouville Problems [Fomin–Gelfand [14, pp.198–205, §41]]
I. Calculus tools for finding extrema of functions: Kaplan [22, §2.19; §2.20].
Tools in calculus of variations for finding extrema of functionals: Direct methods (the Rayleigh–Ritz method; the method of finite differences) and using Euler equations [Courant–Hilbert [10, vol.1, chap. IV, §2]].
II. Solving Sturm–Liouville Problems effectively [Fomin–Gelfand [14, pp.196–197, Remark 2]] by the Ritz method [Fomin–Gelfand [14, p.196, Theorem]]: construct a complete sequence of functions φ_n as in Fomin–Gelfand [14, p.195, (8)]; this sequence allows us to reduce the problem of finding the minimum of the functional $J[y]$ to the problem of finding the minimum of the function $J[\alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n]$ of the n variables $\alpha_1, \cdots, \alpha_n$ [Fomin–Gelfand [14, p.195, (10)]]. Thus, it suffices to calculate y_n given in Fomin–Gelfand [14, p.196, l.13–l.14] by using calculus tools for finding extrema for functions.
III. The existence of $\lambda^{(1)}$ given in Fomin–Gelfand [14, p.200, (24)] is more constructive and effective than the existence of μ_0 given in Coddington–Levinson [7, p.195, l.-9].

Explanation. (A).
1. M defined as in Fomin–Gelfand [14, p.199, l.5] can be computed by calculus.
2. For a system's solution, we may replace its function (uncountable) form $y(x)$ with its sequence (countable) form α_k as in Fomin–Gelfand [14, p.199, (18)]. Thus, $J[y]$ is transformed to $J(\alpha_1\varphi_1 + \cdots + \alpha_n\varphi_n)$, a quadratic form in $\alpha_1, \cdots, \alpha_n$. The minimum of the latter can be computered by the methods given in Kaplan [22, §2.19; §2.20].
3. Define $\lambda_n^{(1)}, y_n^{(1)} \quad (n = 1, 2, \cdots)$ as in [Fomin–Gelfand [14, p.199, l.-10–l.-7]]. Then $\lambda_{n+1}^{(1)} \leq \lambda_n^{(1)}$ [Fomin–Gelfand [14, p.200, (23)]]. Define $\lambda^{(1)}$ as in Fomin–Gelfand [14, p.200, (24)]. After obtaining $\lambda_1^{(1)}, \cdots, \lambda_m^{(1)}$, we know $\lambda^{(1)}$ is between $\lambda_m^{(1)}$ and the lower bound of $\{\lambda_n^{(1)}\}$. Thus, the possible range of $\lambda^{(1)}$ is getting shorter and shorter as the process goes on. In Fomin–Gelfand [14, p.201, l.-14–p.203,

l.-3], we use the method of Lagrange multipliers to obtain Fomin–Gelfand [14, p.203, (36)] and then use Fomin–Gelfand [14, p.201, Lemma 2] to prove Fomin–Gelfand [14, p.202, (32)].
(B). In contrast, $\mu_0 = \sup_{\|u\|=1} |(\mathscr{G}u, u)| \quad (u \in C \text{ on } [a,b])$ [Coddington–Levinson [7, p.195, l.2; l.-9]]. The existence of supremum is derived from reduction to absurdity [Rudin [29, p.11, l.-17–l.-16]]. We have no way to know its location on the real line. Furthermore, as we collect more elements of the index set $(u \in I)$ and find $\sup\{(\mathscr{G}u, u) | u \in I\}$, this procedure will not help narrow down the search scope of the final supremum.
Remark. Based on (A), one can easily create a effective computer program to find $\lambda^{(1)}$. However, the idea given in (B) is useless for one to find μ_0 using a computer. Mathematicians should put more effective stuff than the content given in Coddington–Levinson [7, p.194, l.-6–p.197, l.8] into mathematical textbooks.
IV. By III, $\lambda^{(1)}, \lambda^{(2)}, \cdots; y^{(1)}, y^{(2)}, \cdots$ [Fomin–Gelfand [14, §41.4)]] can be effectively calculated using the method of Lagrange multipliers, while the existence of $\mu_k \quad (k = 0, 1, 2, \cdots)$ given in Coddington–Levinson [7, p.195, l.-9–p.196, l.-2] is derived from the $(k+1)$th level of reduction to absurdity. Furthermore, that the process of finding $\mu_0, \mu_1, \cdots$ can be continued is proved by reduction to absurdity [Coddington–Levinson [7, p.197, l.1–l.7]], while that the process of constructing $\lambda^{(1)}, \lambda^{(2)}, \cdots$ can be continued because each step of the process satisfies the conditions of the method of Lagrange multipliers. □

(7) Suppose we want to prove the equality of two functions with parameters. We first find the two differential equations that they satisfy and then manage to transform one of these differential equations to the other. This is the most powerful and effective method in analysis. Suppose we want to prove $\cos(nz) = {}_2F_1(\frac{1}{2}n, -\frac{1}{2}n; \frac{1}{2}; \sin^2 z)$ [Sneddon [32, p.40, Problem 2(i)]]. When n is a positive even integer, we may express $\cos(nz)$ as a polynomial of $\sin^2 z$ and then use combinatory analysis to prove $\cos(nz) = {}_2F_1(\frac{1}{2}n, -\frac{1}{2}n; \frac{1}{2}; \sin^2 z)$. The proof is not easy. Even if we succeed in proving these special cases, to extend the equality from the case of positive even integers to the case of complex numbers is still a big problem to be solved. The method of mathematical induction and all the methods in complex analysis are not competent enough for this task. Whipple's formula [Guo–Wang [16, p.293, Problem 57]] is another example.

(8) $W_{k,m}(z) = -\frac{1}{2\pi i}\Gamma(k+\frac{1}{2}-m)e^{-z/2}z^k \int_\infty^{(0+)} (-t)^{-k-1/2+m}(1+\frac{t}{z})^{k-1/2+m}e^{-t}dt$ [Watson–Whittaker [40, p.339, l.-13–l.-12]] follows from Watson–Whittaker [40, p.292, l.-15–l.-10] and Guo–Wang [16, p.95, l.-8].
Remark. (Methodical solutions) The differential equation given in Watson–Whittaker [40, p.291, l.-11–l.-7] belongs to a special type. The given solution is justified simply by substitution [Watson–Whittaker [40, p.292, l.-15–l.-10]]. We do not know from where the integrand comes. The underdeveloped solution based on guess, luck, and trial-and-error such as Watson–Whittaker [40, p.339, l.-13–l.-12] cannot be considered a methodical solution. In contrast, the integral solution given in Guo–Wang [16, p.305, l.10–l.19; §6.4] is built by a systematic method which applies to the wider class of equations of Laplacian type [Guo–Wang [16, §2.13]]. In fact, the integrand and the path of integration [Guo–Wang [16, p.302, l.4–l.13]] can be specified by the Laplace transform. Consequently, the latter solution is more methodical than the former one.

(9) (Applications of analytic continuation to the Weber–Schafheitlin integral: the right timing for a statement's appearance)

Suppose we choose the weakest possible conditions required in an argument to be our theorem's hypothesis. If the argument has used the method of analytic continuation [Rudin [31, §16.9–§16.16]] no

more than once, then no confusion will occur. However, what should we do if the argument has used the method of analytic continuation more than once? Let us see the following example.
Example. Let $A(z)=\sum_{m=0}^{\infty}\frac{(-)^m(a/2)^{\alpha-\beta+\gamma+2m-1}\Gamma(2\alpha+2m)\Gamma(\alpha-\beta+\gamma+2m)}{z^{2\alpha+2m}m!\Gamma(\alpha-\beta+m+1)\Gamma(\gamma+m)\Gamma(\alpha-\beta+\gamma+m)}$;
$B(z)=\int_0^\infty\frac{J_\mu(at)J_\nu(at)}{t^{\gamma-\alpha-\beta}}dt$, $D_1=\{z|\Re(z)>2a\}$;
$C(z)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\frac{(a/2)^{\alpha-\beta+\gamma+2s-1}\Gamma(2\alpha+2s)\Gamma(\alpha-\beta+\gamma+2s)}{z^{2\alpha+2s}\Gamma(\alpha-\beta+s+1)\Gamma(\gamma+s)\Gamma(\alpha-\beta+\gamma+s)}\Gamma(-s)ds$, $D_2=\{z||\arg z|<\pi\}$.
Watson [41, p.402, l.13–l.19] shows that (B,D_1) is an analytic continuation of A; Watson [41, p.402, l.-10–l.-4] shows that (C,D_2) is ananalytic continuation of A. Since $D_1\subset D_2$, we can say that (B,D_2) is an analytic continuation of A. In order to establish the first analytic continuation, we must impose the condition $z\in D_1$. After establishing the second analytic continuation, we find that the condition $z\in D_1$ can be weakened to the condition $z\in D_2$. However, before we establish the second analytic continuation, there is no way to know that (B,D_2) is an analytic continuation of A. Thus, the paragraph given in Watson [41, p.402, l.-13–l.-11] has the problem with timing; we should collect enough evidence before we propose a hypothesus. Therefore, whenever we use the method of analytic contiuation, we should check and record if the change of the condition is needed so that we may easily clarify the relationship between cause and effect in the proof structure.

In fact, Watson [41, §13·4; §13·41] are self-contained, but its author has written the facts in the form of previews because of the timing problem. Every time he says that a condition ensures convergence, the readers may not be able to prove the fact at that moment, but they should be able to find the proof later in the section if they are patient enough. However, some impatient readers may think that they must find the proof somewhere else. The incorrect claim given in Guo–Wang [16, p.405, l.7–l.9] is sufficient to show that there are many people under the mistaken impression. In fact, one cannot see the convergence of $\int_0^\infty\frac{J_\mu(at)J_\nu(bt)}{t^\lambda}dt$ [Watson [41, p.399, l.2–l.5]] until one reads up to Watson [41, p.401, l.15]]. Similarly, one cannot see the convergence of $\int_0^\infty\frac{J_\mu(at)J_\nu(at)}{t^\lambda}dt(\mu-\nu$ is an odd integer$;0>\Re(\lambda)>-1)$ [Watson [41, p.403, l.-8]] until one reads up to Watson [41, p.404, (3)]].
Remark 1. $\sum_{m=0}^{\infty}\frac{(-)^m(z/2)^{\gamma+2m-1}}{m!\Gamma(\gamma+m)}|\int_0^\infty e^{-ct}J_{\alpha-\beta}(at)t^{\alpha+\beta+2m-1}dt|$ is absolutely convergent when $|z|<c$ [Watson [41, p.399, l.-9–l.-8]].

Proof. By Watson [41, p.385, (2)], $\int_0^\infty e^{-ct}J_{\alpha-\beta}(at)t^{\alpha+\beta+2m-1}dt=O(c^{-(2\alpha+2m)})$. Then use the ratio test. □

Remark 2. We impose the condition $\Re z>0$ [Watson [41, p.399, l.18]] because $z^{\gamma+2m+1}$ [Watson [41, p.399, l.20]] requires the consideration of the domain of $\log z$. We impose the condition $|\Im z|<c$ to ensure the convergence of $\int^\infty e^{-ct}J_{\gamma-1}(zt)t^{-\lambda}dt$; see Jackson [19, p.114, (3.91)].
Let $D=\{z|\Re z>0,|z|<c\}$, $D'=\{z|\Re z>0,|\Im z|<c,|z|<\sqrt{a^2+c^2}-c\}$ and
$f(z)=\int_0^\infty e^{-ct}\frac{J_{\alpha-\beta}(at)J_{\gamma-1}(zt)}{t^{\gamma-\alpha-\beta}}dt=\sum_{m=0}^{\infty}\frac{(-)^m(z/2)^{\gamma+2m-1}}{m!\Gamma(\gamma+m)}\cdot\frac{(a/2)^{\alpha-\beta}\Gamma(2\alpha+2m)}{(a^2+c^2)^{\alpha+m}\Gamma(\alpha-\beta+1)}{}_2F_1(\alpha+m,1/2-\beta-m;\alpha-\beta+1;\frac{a^2}{a^2+c^2})$.
Watson [41, p.399, l.-14–l.-5] shows that $f(z)$ is analytic on D. Watson [41, p.400, l.1–l.23] shows that (f,D') is an analytic continuation of (f,D). In order to prove the analyticity of f on D, we impose the condition $|z|<c$; after the establishment of the analytic continuation (f,D'), we find that the condition $|z|<c$ can be weakened to the condition $|z|<\sqrt{a^2+c^2}-c$. Thus, using the method of analytic continuation is like ascending to a higher floor: our views become broader and farther.
Remark 3. By Rudin [29, p.135, Theorem 7.11], the limit of the series when $c\to 0$ is the same as the value of the series when $c=0$ [Watson [41, p.401, l.8–l.9]]. “Provided that the integral is convergent”

[Watson [41, p.401, l.-6]] means "the condition given in Watson [41, p.399, l.3] is satisfied".
Remark 4. By Jackson [19, p.114, (3.91)], $J_{\alpha-\beta}(at), J_{\gamma-1}(at) = O(e^{at})$. In order to ensure the convergence of $\int^{\infty} e^{-zt} J_{\alpha-\beta}(at) J_{\gamma-1}(at) dt$, we impose the condition $\Re z > 2a$ [Watson [41, p.402, l.-20]].
Remark 5. If $\Re z > 0$ and $|z| < 2a$, then
$\int_0^{\infty} \frac{J_{\alpha-\beta}(at)J_{\gamma-1}(at)}{t^{\gamma-\alpha-\beta}} dt = \frac{1}{2}\sum_{m=0}^{\infty} \frac{(-)^m (a/2)^{\gamma-\alpha-\beta-m-1} z^m \Gamma(\gamma-\alpha-\beta-m)\Gamma(\alpha+m/2)}{m!\Gamma(1-\beta-m/2)\Gamma(\gamma-\alpha-m/2)\Gamma(\gamma-\beta-m/2)}$
$+ \frac{1}{2}\sum_{m=0}^{\infty} \frac{(-)^m (a/2)^{-m-1} z^{\gamma-\alpha-\beta+m}\Gamma(\alpha+\beta-\gamma-m)\Gamma(\alpha/2-\beta/2+\gamma/2-m/2)}{m!\Gamma(\alpha/2-\beta/2-\gamma/2-m/2+1)\Gamma(\beta/2+\gamma/2-\alpha/2-m/2)\Gamma(\alpha/2-\beta/2+\gamma/2-m/2)}$ [Watson [41, p.403, l.3–l.5]].

Proof. We choose Watson–Whittaker [40, p.288, l.-16–p.289, l.5] or Guo–Wang [16, p.154, Fig. 9] to be our primitive model for development. By Guo–Wang [16, p.100, (8)], $\Gamma(2\alpha+2s)$ provides the factor 2^{2s}, so does $\Gamma(\alpha-\beta+\gamma+2s)$. The numerator of the integrand given in Watson [41, p.402, l.-8] provides the factor $(a/2)^{2s}$. Consequently, instead of considering $|(-z)^s \csc s\pi|$ [Guo–Wang [16, p.155, l.10]], we should consider $|(\frac{4a}{2z})^{2s}\frac{1}{\sin s\pi}|$
$= O(\exp[(N+1/2)\cos\theta \ln|\frac{4a}{2z}|^2 - (N+1/2)\delta|\sin\theta|])$ ($\ln|\frac{2a}{z}| > 0$ because $|z| < 2a$)
$$= \begin{cases} O(\exp[-2^{-1/2}(N+1/2)\ln|\frac{4a}{2z}|^2]) & \text{if } -\pi < \theta \le -3\pi/4 \text{ or } 3\pi/4 \le \theta < \pi \\ O(\exp[-2^{-1/2}(N+1/2)]) & \text{if } -3\pi/4 \le \theta \le -\pi/2 \text{ or } \pi/2 \le \theta \le 3\pi/4 \end{cases}. \qquad \square$$

Remark 6. $z^{\gamma-\alpha-\beta} = e^{(\gamma-\alpha-\beta)\ln z}$.
$|z^{\gamma-\alpha-\beta}| = e^{\Re(\gamma-\alpha-\beta)\ln|z| - \arg z \cdot \Im(\gamma-\alpha-\beta)}$.
If $\Re(\gamma-\alpha-\beta) > 0$ and $z = c \to 0$, then $|z^{\gamma-\alpha-\beta}| = e^{\Re(\gamma-\alpha-\beta)\ln c} \to 0$ [Watson [41, p.403, (1)]].
Remark 7. "It is supposed that these relations hold down to the end of §13·41." [Watson [41, p.399, l.12]] should have been corrected as follows:
"In Watson [41, p.399, l.7–p.403, l.-9], $(\mu,\nu,\lambda) \leftrightarrow (\alpha,\beta,\gamma)$ is transfo rmed according to the relations given in Watson [41, p.399, l.9–l.11]; $\alpha = (\mu+\nu-\lambda+1)/2$. In Watson [41, p.403, l.-8–p.404, l.-7], $(\mu,\nu,\lambda) \leftrightarrow (\alpha,p,\lambda)$ is transformed according to the relations given in Watson [41, p.403, l.-6–l.-5]; $\alpha = (\mu+\nu+1)/2$."
It is really confusing to use the same notation α in the same section [Watson [41, §13·41]] to represent two different quantities. The latter α should have been replaced with another notation, for example, η.
Remark 8. Without loss of generality we may assume that $p = 0, 1, 2, \cdots$ [Watson [41, p.403, l.-8; l.-6–l.-5]].
Remark 9. Since $\Re\lambda < 0$, by Bromwich [5, p.203, l.-7–l.-5; p.204, l.-17–l.-15], both ${}_2F_1(\alpha-\frac{\lambda}{2}, -p-\frac{\lambda}{2}; \alpha-p; 1)$ and ${}_2F_1(\alpha-\frac{\lambda}{2}, p+1-\frac{\lambda}{2}; \alpha+p+1; 1)$ diverge [Watson [41, p.404, l.10–l.11]].
The following supplements may help us understand the proof of the theorem given in Bromwich [5, p.204, l.13–l.22]:
(1). In order to obtain $\frac{a_n}{a_{n+1}} < 1 + \frac{2}{n}$ [Bromwich [5, p.34, l.-8]], we must impose the condition that σ_n's are bounded.
(2). $\sum a_n$ converges $\Leftrightarrow \lim(na_n) = 0$ [Bromwich [5, p.35, l.16–l.17]].

Proof. I. $\frac{na_n}{(n+1)a_{n+1}} = 1 + \frac{1}{n}[(\mu-1)\frac{n}{n+1} + \frac{n}{n+1}\frac{\omega_n}{n^{\lambda-1}}]$.
II. $\Rightarrow$:
By Bromwich [5, p.35, l.12], $\mu > 1$.
By I and Bromwich [5, Art. 39, Ex. 3], $\lim(na_n) = 0$.
$\Leftarrow$:
By Bromwich [5, p.35, l.12], $\mu \le 1$.
Case $\mu < 1$: By I, $\frac{na_n}{(n+1)a_{n+1}} \le 1$. Hence, $na_n \nearrow$.

Case $\mu = 1$: By induction, $\sum_{m=1}^{n} a_m = O(na_n)$.
If $\sum a_n$ diverges, then there exists an subsequence n_k such that $\sum_{m=1}^{n_k} a_m \to L \neq 0$.
$\exists M > 0$: $|L(n_k a_{n_k})^{-1}| \leq M$. This contradicts $\lim(na_n) = 0$. $\square$

(3). By Rudin [29, p.62, Theorem 3.43], the hypergeometric series given in Bromwich [5, p.35, l.-16] converges for $x = -1$, if $\gamma + 1 > \alpha + \beta$.
(4). Without imposing proper conditions, the three theorems given in Bromwich [5, p.201, l.4; l.5; l.-10] cannot be valid. However, our goal is proving the theorem given in Bromwich [5, p.204, l.13–l.22]. Consequently, all we have to do is impose some conditions so that the above three theorems are valid for the cases (1), (2), and (3) given in Bromwich [5, p.204, l.-3]. For example, the theorem given in Bromwich [5, p.201, l.5] is valid for case (3) because $|\frac{a_n}{a_{n+1}}| + \frac{D_{n+1}}{D_n} > \frac{1}{2}$ ($|\frac{a_n}{a_{n+1}}| \to 1$ and $\frac{D_{n+1}}{D_n} \to 0$ as $n \to \infty$). The proof of $\underline{\lim}\, \kappa_n > 0$ [Bromwich [5, p.201, l.-10]] can be proved as follows:

Proof. $\underline{\lim}[f(n)(1 + 2\frac{f'(n)}{f(n)} + \frac{\kappa_n}{f(n)}) - \frac{f^2(n+1)}{f(n)}] > 0$ [Bromwich [5, p.201, l.4]].
$f(n)^2 + 2f(n)f'(n) - f^2(n+1)$
$= -2\int_0^1 [f(n+x)f'(n+x) - f(n)f'(n)]dx$
$= -2\int_0^1 \frac{d}{dt}[f(n+t)f'(n+t)]dt$
$= -2\int_0^1 f(n+t)(\frac{f'^2(n+t)}{f(n+t)} + f''(n+t))dt$.
For cases (1), (2), and (3), $|\frac{f(n+t)}{f(n)}| \leq 1$. $f''(n+t), \frac{f'^2(n+t)}{f(n+t)} \to 0$ [Bromwich [5, p.201, l.-5]] as $n \to \infty$. $\square$

(5). "$\underline{\lim}(\log n)[n\{|\frac{a_n}{a_{n+1}}|^2 - 1\} - 2]] > 2$ (convergence); $\overline{\lim}(\log n)[n\{|\frac{a_n}{a_{n+1}}|^2 - 1\} - 2]] < 2$ (divergence)" [Bromwich [5, p.202, l.5, (3)]] should have been corrected as "$\underline{\lim}(\log n)[n\{|\frac{a_n}{a_{n+1}}|^2 - 1\} - 2]] > 0$ (convergence); $\overline{\lim}(\log n)[n\{|\frac{a_n}{a_{n+1}}|^2 - 1\} - 2]] < 0$ (divergence)".
(6). If $\alpha = 0$, then $|a_m| \to L > 0$ as $m \to \infty$ [Bromwich [5, p.203, l.3–l.5]].

Proof. $|\frac{a_n}{a_{n+1}}|^2 = 1 + \frac{\omega}{n^\lambda}$.
$1 - \varepsilon n^{\delta-\lambda} \leq |\frac{a_n}{a_{n+1}}|^2 \leq 1 + \varepsilon n^{\delta-\lambda}$. Consequently,
$1 - \varepsilon \sum_{k=n}^{m} k^{\delta-\lambda} \leq \sum_{k=n}^{m} |\frac{a_n}{a_{n+1}}|^2$ [Bromwich [5, p.95, l.-9]]
$\leq 1 + \varepsilon \sum_{k=n}^{m} k^{\delta-\lambda}$. $\square$

(7). If $\mu = 1$, then $\sum a_n$ diverges [Bromwich [5, p.204, l.-7–l.-6]].

Proof. Assume that $\sum a_n$ converges to a number L.
By induction, $\sum_{m=1}^{n} a_m = O(na_n)$.
$\exists M > 0$: $|\frac{L}{na_n}| \leq M$.
Since $L/n \to 0$, $a_n \not\to 0$. $\square$

(8). By induction, the sum of n terms of this series is $\frac{(2-\mu)(3-\mu)\cdots(n-\mu)}{1\cdot 2\cdot 3\cdots(n-1)}$ [Bromwich [5, p.204, l.-4–l.-3]].
(9). If $0 < \alpha \leq 1$, then $1 + \frac{1-\mu}{1} + \frac{(1-\mu)(2-\mu)}{1\cdot 2} + \frac{(1-\mu)(2-\mu)(3-\mu)}{1\cdot 2\cdot 3} + \cdots$ diverges [Bromwich [5, p.204, l.-2–l.-1]].

Proof. Case $0<\alpha<1$: $|1+\frac{1-\mu}{n-1}|^2 \geq 1+\frac{2(1-\alpha)}{n-1}$.
Case $\alpha=1$: $\arg(1-\frac{i\beta}{m}) = -\arcsin\frac{\beta}{m} \approx -\frac{\beta}{m}$.
Consequently, $\arg[\Pi_{n+1}^{\infty}(1-\frac{i\beta}{m})] \approx -\sum_{m=n+1}^{\infty}\frac{\beta}{m} = -\beta\cdot\infty$. □

Remark 10. Since $\lim_{s\to-\alpha}\frac{\Gamma(2\alpha+2s)}{\Gamma(\alpha+s-p)} = \frac{1}{2}\lim_{s\to-\alpha}\frac{2(\alpha+s)\Gamma(2\alpha+2s)}{(s+\alpha)\Gamma(\alpha+s-p)}$, the residue at $s=-\alpha$ is $(-)^p/(2a)$ [Watson [41, p.404, l.−14]].
Remark 11. By Watson [41, p.403, (2), ($\lambda=1$)] and Guo–Wang [16, p.94, (1); p.99, (3)], $\int_0^{\infty}\frac{J_\mu(at)J_\nu(at)}{t}dt = \frac{2}{\pi}\frac{\sin[(\nu-\mu)\pi/2]}{\nu^2-\mu^2}$ [Watson [41, p.404, l.−5]].

(10) (Integration on a Riemann surface with branch points)

If we reduce a contour integral on a Riemann surface to an integral along a line segment, the value of the latter integral may depend on which sheet the line segment is in, while the former integral is an invariant quantity. When we reduce a contour integral on a Riemann surface to an integral along a line segment, we often have to degeneate a part of the contour to a point. In order to make the argument of points along the contour continuous and simplify the calculation of these arguments, we should restore the degeneated point to its corresponding nondegeneate part. For example, in order to prove Watson [41, p.168, (3)], we must prove that
$\int_{\infty\exp i\beta}^{(0+)} e^{-u}(-u)^{\nu-1/2}(1+\frac{iu}{2z})^{\nu-1/2}du = [e^{-i\pi(\nu-1/2)}-e^{i\pi(\nu-1/2)}]\int_0^{\infty\exp i\beta} e^{-u}u^{\nu-1/2}(1+\frac{iu}{2z})^{\nu-1/2}du.$

Proof. Let $I=\infty\exp i\beta, A=\delta e^{i(\beta-2\pi)}, B=\delta e^{i(\beta-\pi)}, C=\delta e^{i\beta}$; IA and CI be line segments; AB and BC are counterclockwise half-circles.
Note that IA and CI are on different sheets.
$\int_{\infty\exp i\beta}^{(0+)} = \int_{IAB}+\int_{BCI}$.
We take the argument of $-u$ in the range between $\beta-2\pi$ and β.
$\int_{BCI} = \int_0^{\infty\exp i\beta} e^{-u}(-u)^{\nu-1/2}(1+\frac{iu}{2z})^{\nu-1/2}du$
$=(e^{-\pi i})^{\nu-1/2}\int_0^{\infty\exp i\beta} e^{-u}u^{\nu-1/2}(1+\frac{iu}{2z})^{\nu-1/2}du.$
$\int_{IAB} = \int_{\infty\exp i\beta}^{0} e^{-u}(-u)^{\nu-1/2}(1+\frac{iu}{2z})^{\nu-1/2}du$
$=(e^{\pi i})^{\nu-1/2}\int_{\infty\exp i\beta}^{0} e^{-u}u^{\nu-1/2}(1+\frac{iu}{2z})^{\nu-1/2}du$
$=-(e^{\pi i})^{\nu-1/2}\int_0^{\infty\exp i\beta} e^{-u}u^{\nu-1/2}(1+\frac{iu}{2z})^{\nu-1/2}du.$
The ending point of the integration path $[\infty\exp i\beta, 0]$ comes from the ending point of the integration path IAB, namely, B. So the argument of u at the $u=0$ is $\beta-\pi$. Then the argument of $-u$ at the $u=0$ is β. Thus, $\beta-(\beta-\pi)=\pi$. □

Similarly, in order to prove Guo–Wang [16, p.371, (11)], we must prove the equality given in Guo–Wang [16, p.371, l.7–l.8].

Proof. Let $I=1+i\infty, A=1+\delta e^{-3\pi i/2}, B=1+\delta e^{-\pi i}, C=1+\delta e^{-\pi i/2}$; IA and CI be line segments; AB and BC are counterclockwise half-circles.
$\int_{1+i\infty}^{(0+)} = \int_{IAB}+\int_{BCI}$.
Based on the restriction given in Guo–Wang [16, p.371, l.10], at the beginning point of integration path, the argument of $t-1$ is $-3\pi/2$, while the argument of $1-t$ is $-\pi/2$, so
$\int_{IAB} = \int_{1+i\infty}^{1} e^{izt}(t^2-1)^{\nu-1/2}dt = (e^{-\pi i})^{\nu-1/2}\int_{1+i\infty}^{1} e^{izt}(1-t^2)^{\nu-1/2}dt.$
$\int_{BCI} = \int_1^{1+i\infty} e^{izt}(t^2-1)^{\nu-1/2}dt = -\int_{1+i\infty}^{1} e^{izt}(t^2-1)^{\nu-1/2}dt$
$=-(e^{\pi i})^{\nu-1/2}\int_{1+i\infty}^{1} e^{izt}(1-t^2)^{\nu-1/2}dt.$

This is because at the beginning point of the integration path, the argument of $t-1$ is $\pi/2$, while the argument of $1-t$ is $-\pi/2$. $\square$

Remark. The above proof shows that $\int_{1+i\infty}^{(0+)} e^{izt}(t^2-1)^{\nu-1/2}dt = (S-T)U$, where $S=(e^{-\pi i})^{\nu-1/2}, T=(e^{\pi i})^{\nu-1/2}, U=\int_{1+i\infty}^{1} e^{izt}(1-t^2)^{\nu-1/2}dt$. If we remove the restriction given in Guo–Wang [16, p.371, l.10], say, at the beginning point of the integration path in U, we let the argument of $1-t$ be $-5\pi/2$. Then U will add a factor of $(-2\pi i)^{\nu-1/2}$, S will become $e^{\pi i(\nu-1/2)}$, and T will become $e^{3\pi i(\nu-1/2}$. Thus, no matter what value we choose for U, $(S-T)U$ is an invariant quantity.

(11) (Contour integrals for Bessel functions)
Example1. $S_{\nu,\alpha,\beta,\gamma}(\rho,t;a)=\int_0^\infty J_\alpha(\rho x)J_\beta(tx)x^{\gamma+1}dx+\frac{2}{\pi}\sin\frac{(\alpha+\beta+\gamma-2\nu)\pi}{2}K_{\nu,\alpha,\beta,\gamma}(\rho,t;a)$ [Sneddon [34, p.35, l.15–l.16, (2.2.9)]].

Proof. I. Let $C_R=\{Re^{i\theta}|0\leq\theta\leq\frac{\pi}{2}\}$ and $F(z)=\frac{J_\nu(az)+iY_\nu(az)}{J_\nu(az)}J_\alpha(\rho z)J_\beta(tz)z^{1+\gamma}$. We want to prove $\lim_{R\to\infty}\int_{C_R}F(z)dz=0$.

Proof. $(|\cos(az-\frac{\nu\pi}{2}-\frac{\pi}{4})|)^{-1}=2(|e^{i(az-\frac{\nu\pi}{2}-\frac{\pi}{4})}+e^{-i(az-\frac{\nu\pi}{2}-\frac{\pi}{4})}|)^{-1}\leq 4e^{\Re(iaz)}$ (note that $\Re(iz)<0$).
By Guo–Wang [16, pp.378–379], we have
$|H_\nu^{(1)}(az)|\sim\sqrt{\frac{2}{\pi aR}}e^{\Re(iaz)}$ and $|J_\alpha(\rho z)|\leq\sqrt{\frac{2}{\pi aR}}e^{\Re(i\rho z)}$.
We may assume that $\arg(iz)$ lies between $\frac{\pi}{2}+\delta$ and π. $\square$

II. $\int_{[\infty i,0]}F(z)dz=-\int_0^\infty\frac{J_\nu(aiy)+iY_\nu(aiy)}{J_\nu(aiy)}J_\alpha(\rho iy)J_\beta(tiy)(iy)^{1+\gamma}d(iy)$.
$J_\nu(aiy)+iY_\nu(aiy)=H_\nu^{(1)}(aiy)$ [Watson [41, p.73, (1)]]
$=\frac{2}{\pi}K_\nu(ay)i^{-\nu-1}$ [Jackson [19, p.116, (3.101)]].
$J_\nu(aiy)=i^\nu I_\nu(ay)$ [Jackson [19, p.116, (3.100)]].
$-\Re(i^{-2\nu+\alpha+\beta+\gamma+1})=-\Re[e^{\pi i/2(-2\nu+\alpha+\beta+\gamma+1)}]$
$-\Re[(\cos[(-2\nu+\alpha+\beta+\gamma)/2]+i\sin[(-2\nu+\alpha+\beta+\gamma)/2])i]$
$=\sin[(-2\nu+\alpha+\beta+\gamma)/2]$.
III. Let $\Gamma=\sum_{s=1}^p\gamma_s$. Then
$\int_\Gamma F(z)dz=-\pi i\sum_{s=1}^p\operatorname{Res}F(\lambda_s)$ [González [15, p.683, Lemma 9.4]].
$Y_\nu(a\lambda_s)=\frac{-2}{\pi a\lambda_s J_\nu'(a\lambda_s)}$ [Watson [41, p.76, l.2–l.3]]
$=\frac{2}{\pi a\lambda_s J_{\nu+1}(a\lambda_s)}$ [Watson [41, p.45, (4)]].
$\lim_{p\to\infty}\int_\Gamma\Re F(z)dz=-S_{\nu,\alpha,\beta,\gamma}(\rho,t;a)$ [Guo–Wang [16, p.422, l.4–l.10]].
IV. The desired result follows from Watson [41, p.482, l.4–l.5] and Cauchy's theorem. $\square$

Example 2. $S^*_{\nu,H,\beta,\gamma,\delta}=\int_0^\infty J_\beta(ux)J_\gamma(vx)x^{\delta+1}dx+\frac{2}{\pi}\sin\frac{(\delta+\beta+\gamma-2\nu)\pi}{2}K^*_{\nu,H,\beta,\gamma,\delta}(u,v)$ [Sneddon [34, p.35, l.−4–l.−3, (2.2.10)]].

Proof. I. Let $C_R=\{Re^{i\theta}|0\leq\theta\leq\frac{\pi}{2}\}$ and $F(z)=\phi(z)J_\beta(uz)J_\gamma(vz)z^{\delta+1}$, where
$\phi(z)=\frac{z\{J_\nu'(z)+iY_\nu'(z)\}+H\{J_\nu(z)+iY_\nu(z)\}}{zJ_\nu'(z)+HJ_\nu(z)}$. Then $\lim_{R\to\infty}\int_{C_R}F(z)dz=0$.
II. $\int_{[\infty i,0]}\Re F(z)dz=\frac{2}{\pi}\sin\frac{(\delta+\beta+\gamma-2\nu)\pi}{2}K^*_{\nu,H,\beta,\gamma,\delta}(u,v)$.
III. Let $\Gamma=\sum_{s=1}^p\gamma_s$. Then
$\int_\Gamma F(z)dz=-\pi i\sum_{s=1}^p\operatorname{Res}F(\mu_s)$ [González [15, p.683, Lemma 9.4]].
$\int_\Gamma\Re F(z)dz=-2\sum_{s=1}^p\frac{J_\beta(u\mu_s)J_\gamma(v\mu_s)}{(\mu_s^2-\nu^2+H^2)J_\nu^2(\mu_s)}\mu_s^{2+\delta}(0<u<1,0<v<1)$ [Watson [41, p.480, l.21–l.24]].

Proof. $\frac{d}{dz}(zJ'_\nu(z)+HJ_\nu(z))|_{z=\mu_s} = -\frac{H^2+\mu^2-\nu^2}{\mu_s}J_\nu(\mu_s)$.
$\begin{vmatrix} zJ'_\nu(z)+HJ_\nu(z) & zY'_\nu(z)+HY_\nu(z) \\ \frac{d}{dz}(zJ'_\nu(z)+HJ_\nu(z)) & \frac{d}{dz}(zY'_\nu(z)+HY_\nu(z)) \end{vmatrix} = z^2\begin{vmatrix} J'_\nu(z) & Y'_\nu(z) \\ J''_\nu(z) & Y''_\nu(z) \end{vmatrix}$
$+Hz(\begin{vmatrix} J'_\nu(z) & Y_\nu(z) \\ J''_\nu(z) & Y'_\nu(z) \end{vmatrix} + \begin{vmatrix} J_\nu(z) & Y'_\nu(z) \\ J'_\nu(z) & Y''_\nu(z) \end{vmatrix})$
$+(H^2+H)\begin{vmatrix} J_\nu(z) & Y_\nu(z) \\ J'_\nu(z) & Y'_\nu(z) \end{vmatrix} = \frac{2(z^2-\nu^2+H^2)}{\pi z}$ [Watson [41, p.76, (1), (5), & (6)]]. □

IV. The desired result follows from Watson [41, p.482, l.−21–l.−19] [we must assume that $\nu \geq H$] and Cauchy's theorem. □

(12) (Carlini's solution of the Bessel equation for functions of large order)
Watson [41, p.6, l.6–p.7, l.−15 & §8.11] give Carlini's solution of the Bessel equation for functions of large order. The motive of writing u as $nu_0+u_1+u_2/n+\cdots$ [Watson [41, p.7, l.10]] is given in Watson [41, p.7, l.6–p.8]. The following passage explains why we transform A_n to u:
Let $A_n = e^{\int ud\varepsilon}$ and $L_1(A_n)=0$ be a second order differential equation. Then $L_1(A_n) = e^{\int ud\varepsilon}L_2(u)$, where $L_2(u)=0$ is a first order differential equation.

(13) (Evaluation of $\int_0^\infty \sin t\cdot t^{n-1}dt$ by using the Cauchy–Kowalevski theorem)
$\int_0^\infty \sin t\cdot t^{n-1}dt = \Gamma(n)\sin(\frac{1}{2}n\pi)$, where $0<n<1$ [Bromwich [5, p.447, l.10; p.474, l.11]; Watson [41, p.230, l.−10]].

Proof. I. Let $n>0, \xi>0, x=\xi+i\eta$, and $U=\int e^{-xt}t^{n-1}dt$. Then $U(x)=\Gamma(n)/x^n$.

Proof. $\frac{\partial U}{\partial \xi} = -\frac{n}{x}U, \frac{\partial U}{\partial \eta} = -\frac{in}{x}U$.
When $\eta=0$, $U(x)=\Gamma(n)/\xi^n$.
$\Gamma(n)/x^n$ satisfies the above system of partial differential equations and Cauchy data.
By the Cauchy–Kowalevski theorem [John [21, p.74, l.4–l.5]], $U(x)=\Gamma(n)/x^n$ locally along the positive real axis.
By analytic continuation, $U(x)=\Gamma(n)/x^n$ in the half-plane $\xi>0$. □

II. $\int_0^\infty e^{-it}t^{n-1}dt = \lim_{\varepsilon\to 0+}\int_0^\infty e^{-(\varepsilon+i)t}t^{n-1}dt$ [Bromwich [5, p.436, Ex. 2]]
$=\lim_{\varepsilon\to 0+}\frac{\Gamma(n)}{(\varepsilon+i)^n}$ [by I]
$=(\cos\frac{1}{2}\pi - i\sin\frac{1}{2}\pi)\Gamma(n)$. □

(14) The transient solution and the steady state solution of Wangsness [39, p.453, (27-11)]
Let us see what we would miss if we were to study the complementary (transient) solution and the particular (steady state) solution from the viewpoint of ODEs alone:
I. The transient solution
1. The external emf is provided by a battery so that $\mathscr{E}=$ constant [Wangsness [39, p.454, l.1–l.2]].
2. How we obtain the initial conditions from physical considerations [Wangsness [39, p.454, l.17–l.20]].
3. When R is small, $\delta=i\omega_n$ is imaginary, where ω_n is the natural angular frequency [Wangsness [39, p.454, l.−11–l.−5]].
II. The steady state solution
1. Physical preparation [Wangsness [39, p.455, l.2–l.7]].

2. To simplify calculations, we prefer $\mathscr{E}e^{i\omega t}$ to $\mathscr{E}\cos\omega t$ [Wangsness [39, p.455, l.−26–l.−12]].
3. Physical meanings of the solution [Wangsness [39, p.455, l.−6–l.−1]].
4. The physical significance of a complex impedance [Wangsness [39, p.456, l.1–l.2]].
5. The resonance frequency is the angular frequency that the current is in phase with the applied emf [Wangsness [39, p.456, l.−18–l.−14]].
6. Wangsness [39, p.457, Figure 27-6].
7. The relationship between Q and the resonance width $|\Delta\omega|$ [Wangsness [39, p.457, (27-29)]].

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